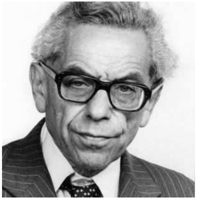
New developments in hypergraph Ramsey theory

Andrew Suk (UC San Diego)

April 24, 2018

Origins of Ramsey theory

"A combinatorial problem in geometry," by Paul Erdős and George Szekeres (1935)



Erdős-Szkeres 1935

Theorem (Monotone subsequence)

Any sequence of $(n-1)^2+1$ *integers contains a monotone subsequence of length* n.

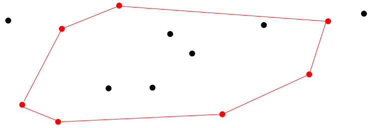
Theorem (Convex polygon)

For any $n > 0$, *there is a minimal* $ES(n)$, *such that every set of* $ES(n)$ *points in the plane in general position contains* n *members in convex position.*

Theorem (Ramsey numbers)

New proof of Ramsey's theorem.

Convex polygon theorem

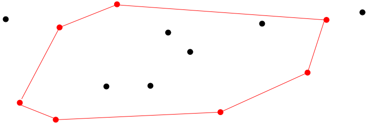


Theorem (Erdős-Szekeres 1935, 1960)

$$2^{n-2} + 1 \leq ES(n) \leq \binom{2n-4}{n-2} + 1 = O(4^n/\sqrt{n}).$$

Conjecture: $ES(n) = 2^{n-2} + 1,\ n \geq 3.$

Convex polygon theorem



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Theorem (S. 2016)

$$ES(n) = 2^{n+o(n)}$$

Erdős-Szekeres 1935

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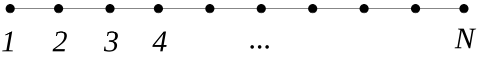
Theorem (Ramsey numbers)

New proof of Ramsey's theorem.

Formal definition: For any integers $k \geq 1$, $s, n \geq k$, there is a minimum $r_k(s, n) = N$, such that for every red/blue coloring of the k-tuples of $\{1, 2, \ldots, N\}$,



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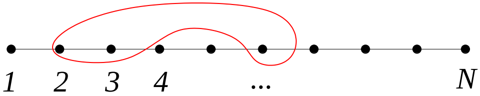

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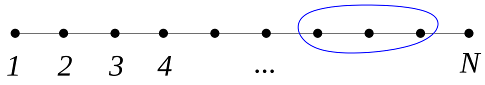

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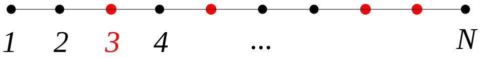

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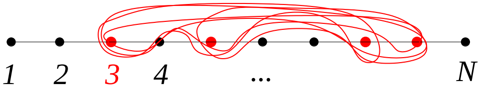

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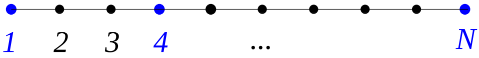

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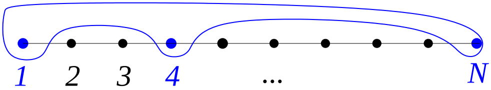

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$r_k(s,n)$ = Ramsey numbers

Graph Ramsey theorem

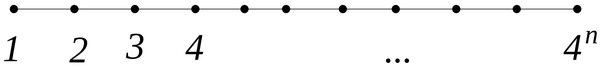
Theorem (Erdős-Szekeres 1935)

$$r_2(s,n) \leq \binom{n+s-2}{s-1}$$

$$r_2(n,n) \leq \binom{2n-2}{n-1} \approx \frac{4^n}{\sqrt{n}}$$

Graph Ramsey number $r_2(n, n) \leq 4^n$

Let G be the complete graph with 4^n vertices, every edge has color red or blue.



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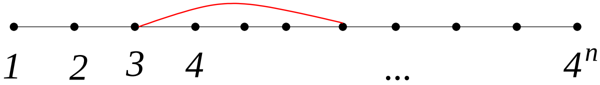

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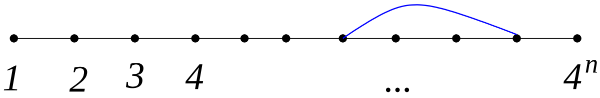

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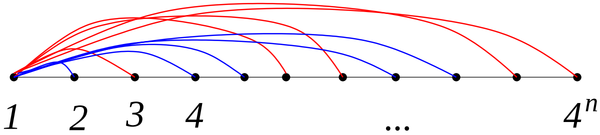

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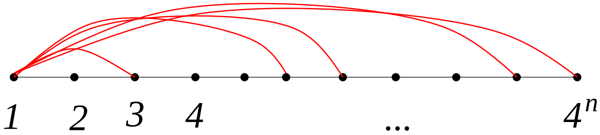

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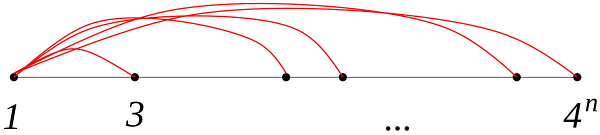

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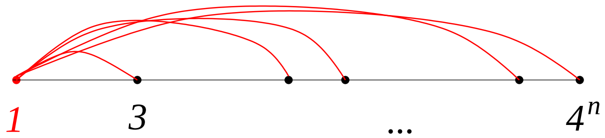

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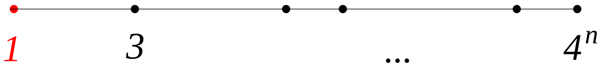

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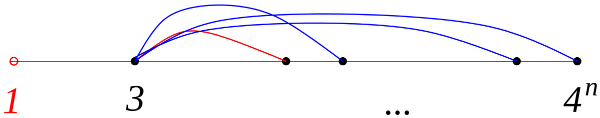

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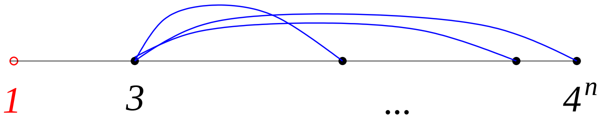

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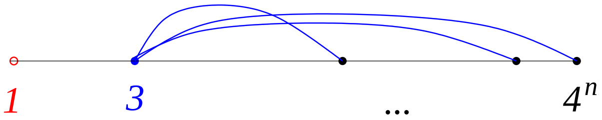

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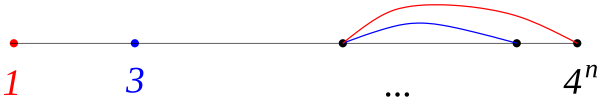

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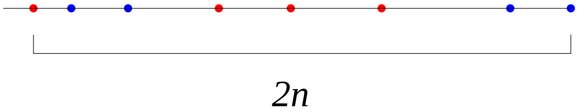

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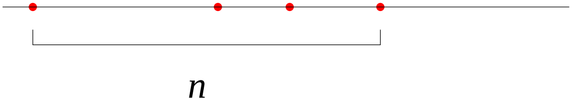

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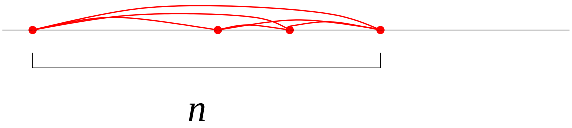


Diagonal graph Ramsey numbers

Theorem (Erdős 1947, Erdős-Szekeres 1935)

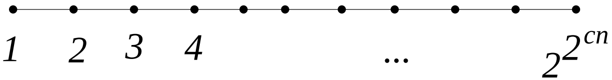
$$(1 + o(1))\frac{n}{e}2^{n/2} < r_2(n, n) < \frac{4^n}{\sqrt{n}}.$$

Theorem (Spencer 1977, Conlon 2008)

$$(1 + o(1))\frac{\sqrt{2}}{e}n2^{n/2} < r_2(n, n) < \frac{4^n}{n^{c \log n / \log \log n}}$$

3-uniform hypergraphs

Greedy argument: $r_3(n,n) \leq 2^{2^{O(n)}}$



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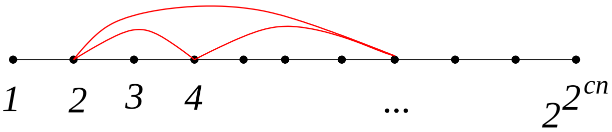

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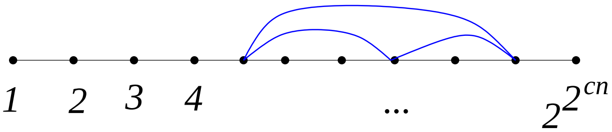

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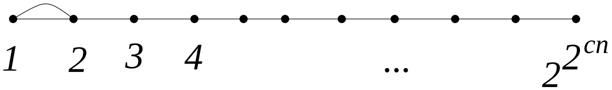

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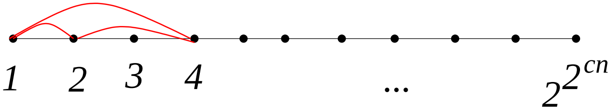

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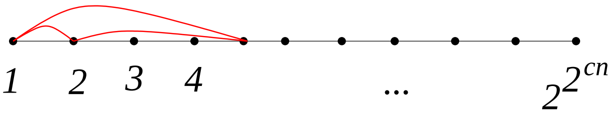

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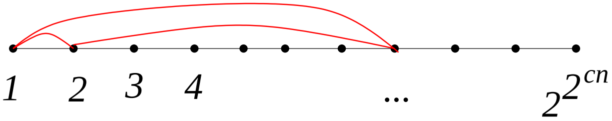

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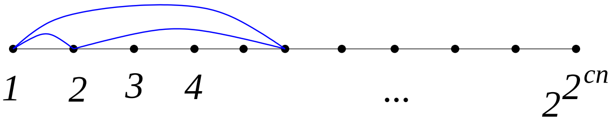

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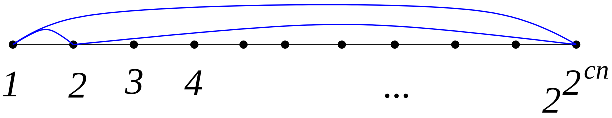

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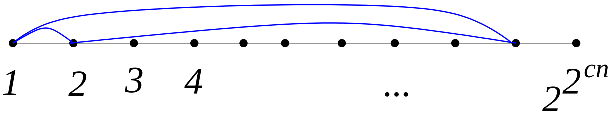

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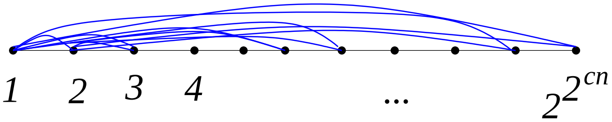

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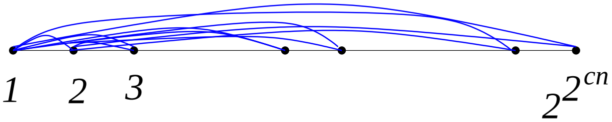

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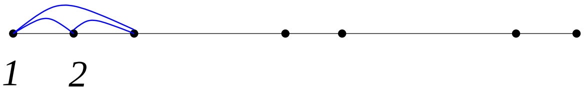

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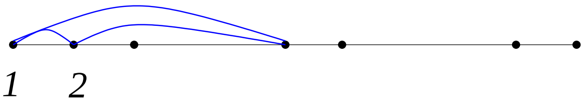

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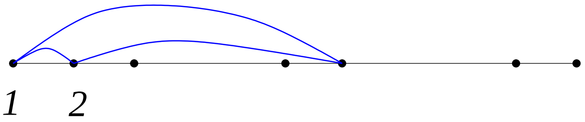

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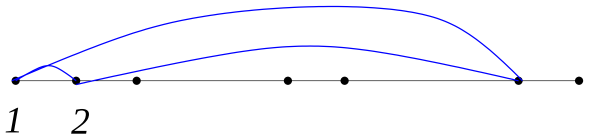

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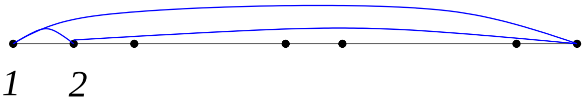

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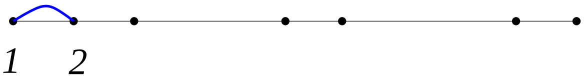

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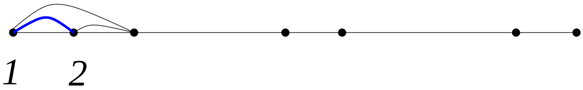

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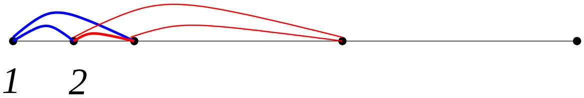

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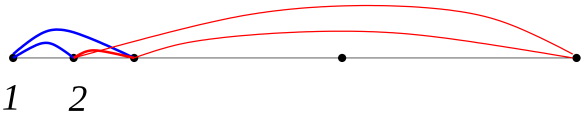

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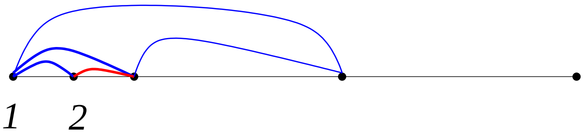

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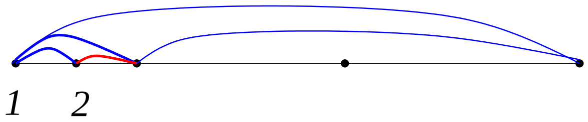

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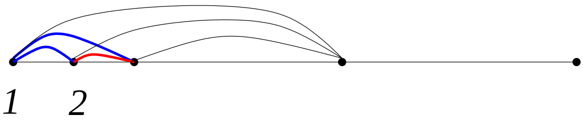

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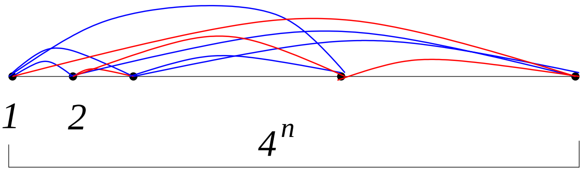

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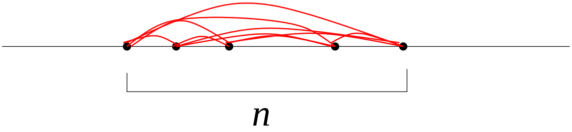

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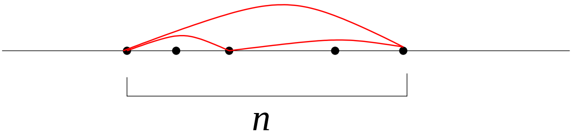


Hypergraph Ramsey numbers

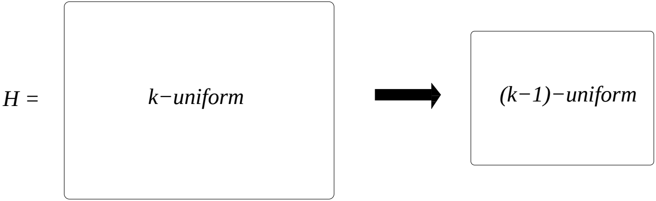
Theorem (Erdős-Rado 1952)

$$2^{cn^2} < r_3(n,n) < 2^{2^{c'n}}$$

Conjecture (Erdős, \$500)

$$r_3(n,n) > 2^{2^{cn}}$$

Erdős-Rado upper bound argument



Greedy argument to reduce the problem to a $(k-1)$-uniform hypergraph problem. Argument shows

$$r_k(s, n) \leq 2^{(r_{k-1}(s-1,n-1))^{k-1}}.$$

Upper bounds for diagonal hypergraph Ramsey numbers

Applying $r_k(s,n) \leq 2^{(r_{k-1}(s-1,n-1))^{k-1}}$.

Conlon (2008): $r_2(n,n) < \frac{4^n}{n^{c\log n/\log\log n}}$

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Erdős-Rado (1952): $r_k(n,n) < \mathrm{twr}_k(cn)$

$\mathrm{twr}_1(x) = x$ and $\mathrm{twr}_{i+1}(x) = 2^{\mathrm{twr}_i(x)}$.

Random constructions

Spencer (1977): $\frac{\sqrt{2}}{e}n2^{n/2} < r_2(n,n) < \frac{4^n}{n^{c\log n/\log\log n}}$

Erdős (1947): $2^{c'n^2} < r_3(n,n) < 2^{2^{cn}}$

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Lower bounds for diagonal hypergraph Ramsey numbers

Erdős-Hajnal stepping up lemma: $k \geq 3$, $r_{k+1}(n,n) > 2^{r_k(n/4,n/4)}$.

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$\vdots$

Erdős-Hajnal: $\mathrm{twr}_{k-1}(c' n^2) < r_k(n,n) < \mathrm{twr}_k(cn)$

$\mathrm{twr}_1(x) = x$ and $\mathrm{twr}_{i+1}(x) = 2^{\mathrm{twr}_i(x)}$.

Conjecture: $r_3(n,n) > 2^{2^{cn}}$

Theorem (Erdős-Hajnal-Rado 1952/1965)

$$2^{cn^2} < r_3(n,n) < 2^{2^{c'n}}$$

Theorem (Erdős-Hajnal)

$$r_3(n,n,n,n) > 2^{2^{cn}}$$

Off-diagonal Ramsey numbers

$r_k(s,n)$ where s is fixed, and $n \to \infty$. $r_k(s,n) \ll r_k(n,n)$

Graphs:

Theorem (Ajtai-Komlós-Szemerédi 1980, Kim 1995)

$r_2(3,n) = \Theta\left(\frac{n^2}{\log n}\right)$

Theorem

For fixed $s > 3$

$$n^{(s+1)/2+o(1)} < r_2(s,n) < n^{s-1+o(1)}$$

$$2^{n/2} < r_2(n,n) < 4^n$$

Upper bounds for off-diagonal Ramsey numbers

3-uniform hypergraphs:

Theorem (Erdős-Hajnal-Rado)

For fixed $s \geq 4$,

$$2^{csn} < r_3(s,n) < 2^{c'n^{2s-4}}.$$

Theorem (Conlon-Fox-Sudakov 2010)

For fixed $s \geq 4$,

$$2^{csn\log n} < r_3(s,n) < 2^{c'n^{s-2}\log n}.$$

$$2^{cn^2} < r_3(n,n) < 2^{2^{c'n}}$$

Fixed $s \geq k+1$.

Erdős-Szekeres (1935): $r_2(s,n) < n^{s-1+o(1)}$

Conlon-Fox-Sudakov (2010): $r_3(s,n) < 2^{cn^{s-2}\log n}$

Upper bounds for off-diagonal hypergraph Ramsey numbers

Fixed $s \geq k + 1$.

Erdős-Szekeres (1935): $r_2(s, n) < n^{s-1+o(1)}$

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Erdős-Rado: $r_k(s, n) < \mathrm{twr}_{k-1}(n^c)$

Lower bounds for off-diagonal hypergraph Ramsey numbers

Fixed $s \geq k+1$.

Bohman-Keevash (2010): $n^{(s+1)/2+o(1)} < r_2(s,n) < n^{s-1+o(1)}$

Conlon-Fox-Sudakov (2010): $2^{c' n \log n} < r_3(s,n) < 2^{cn^{s-2} \log n}$

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Tower growth rate for $r_4(5,n)$ is unknown.

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⋮

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What is the tower growth rate of $r_k(k+1,n)$ and $r_k(k+2,n)$?

Fixed $s \geq k+1$.

Bohman-Keevash (2010): $n^{(s+1)/2+o(1)} < r_2(s,n) < n^{s-1+o(1)}$

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What is the tower growth rate of $r_4(5,n)$ and $r_4(6,n)$?

Towards the Erdős-Hajnal conjecture

Conjecture (Erdos-Hajnal)

$$r_4(5, n), r_4(6, n) > 2^{2^{cn}}$$

Erdős-Hajnal (1972): $r_4(5, n), r_4(6, n) > 2^{cn}$

Mubayi-S. (2017): $r_4(5, n) > 2^{n^2}$ $\quad r_4(6, n) > 2^{n^{c \log n}}$

New lower bounds for off-diagonal hypergraph Ramsey numbers

Theorem (Mubayi-S., 2018)

$$r_4(5,n) > 2^{n^{c\log n}} \qquad r_4(6,n) > 2^{2^{cn^{1/5}}}.$$

for fixed $k > 4$

$$r_k(k+1,n) > \mathrm{twr}_{k-2}(n^{c\log n}) \qquad r_k(k+2,n) > \mathrm{twr}_{k-1}(cn^{1/5}).$$

$$r_k(k+2,n) = \mathrm{twr}_{k-1}(n^{\Theta(1)})$$

Open problems

Diagonal Ramsey problem ($500 Erdős):

$$2^{cn^2} < r_3(n,n) < 2^{2^{cn}}.$$

Off-diagonal Ramsey problem:

$$2^{n^{c \log n}} < r_4(5,n) < 2^{2^{cn}}.$$

Theorem (Mubayi-S. 2017)

Showing $r_3(n,n) > 2^{2^{cn}}$ *implies that* $r_4(5,n) > 2^{2^{c'n}}$.

More off-diagonal?

Off diagonal hypergraph Ramsey numbers: $r_k(k+1, n)$

Red clique size $k+1$ or Blue clique of size n.

$$r_k(k, n) = n \qquad \text{(trivial)}$$

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Off diagonal hypergraph Ramsey numbers: $r_k(k+1, n)$

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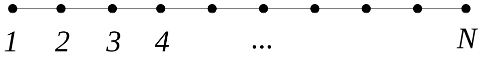
$$r_k(k, n) = n \qquad \text{(trivial)}$$

A more off diagonal Ramsey number:

Almost Red clique size $k+1$ or Blue clique of size n.

Very off-diagonal

Another Ramsey function: Let $r_k(k+1, t; n)$ be the minimum N, such that for every red/blue coloring of the k-tuples of $\{1, 2, \ldots, N\}$,



1. $k+1$ integers which induces at least t red k-tuples, or
2. n integers for which every k-tuple is blue.

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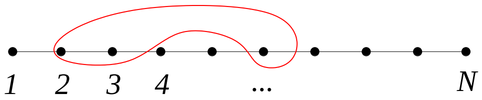

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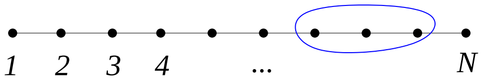

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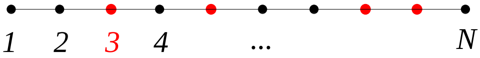

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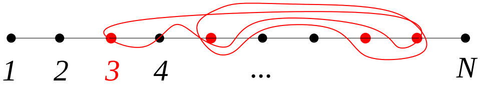
Very off-diagonal

Another Ramsey function: Let $r_k(k+1, {\color{red}t}; n)$ be the minimum N, such that for every red/blue coloring of the k-tuples of $\{1, 2, \ldots, N\}$,



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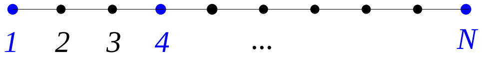

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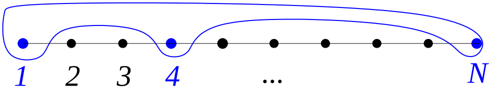

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An old problem of Erdős and Hajnal 1972

Problem (Erdős-Hajnal 1972)

For $k \geq 3$ and $t \in [k+1]$, estimate $r_k(k+1, t; n)$.

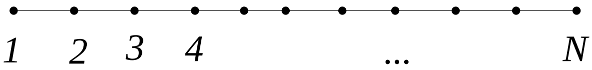
$$r_k(k+1, 1; n) = n$$

$$\vdots$$

$$\mathrm{twr}_{k-2}(n^{c' \log n}) \leq r_k(k+1, k+1; n) = r_k(k+1, n) \leq \mathrm{twr}_{k-1}(n^c)$$

$r_k(k+1, 2; n) < O(n^{k-1}) = N.$

Case 1: $(k-1)$-tuple with 2 red edges.



Known results

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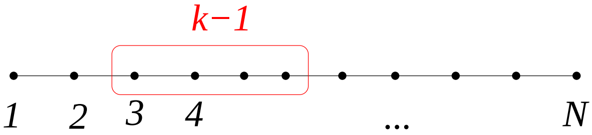

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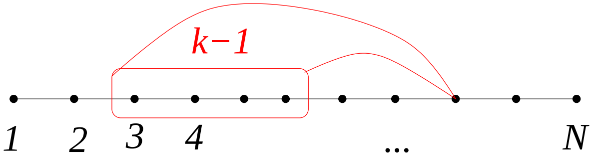

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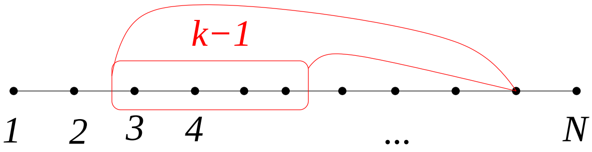

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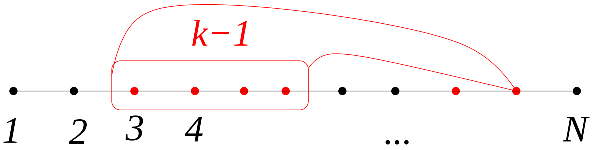

Known results

$r_k(k+1, 2; n) < O(n^{k-1}) = N.$

Case 1: $(k-1)$-tuple with 2 red edges.

Case 2: Less than $\binom{N}{k-1}$ k-tuples are red implies blue copy of $K_n^{(k)}$.

Slightly better bounds

Theorem (Rödl-Šinajová and Kostochka-Mubayi-Verstraete)

$$c\frac{n^{k-1}}{\log n} \leq r_k(k+1, 2; n) \leq c'\frac{n^{k-1}}{\log n}$$

Upper bounds

Erdős-Rado argument: $r_k(k+1, t; n) \leq 2^{(r_{k-1}(k, t-1; n))^{k-1}}$.

Theorem (Erdős-Hajnal 1972)

$$r_k(k+1, 2; n) < cn^{k-1}$$

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Theorem (Erdős-Hajnal 1972)

$$r_k(k+1, 2; n) < cn^{k-1}$$

$$r_k(k+1, 3; n) < 2^{n^c}$$

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$$\vdots$$

$$r_k(k+1, t; n) < \mathrm{twr}_{t-1}(n^c)$$

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$$\vdots$$

$$r_k(k+1, k; n) < \mathrm{twr}_{k-1}(n^c)$$

$$r_k(k+1, k+1; n) < \mathrm{twr}_{k-1}(n^c)$$

Upper bounds

Erdős-Rado argument: $r_k(k+1, t; n) \leq 2^{(r_{k-1}(k, t-1; n))^{k-1}}$.

Theorem (Erdős-Hajnal 1972)

$$r_k(k+1, 2; n) < cn^{k-1}$$

$$r_k(k+1, 3; n) < 2^{n^c}$$

$$r_k(k+1, 4; n) < 2^{2^{n^c}}$$

$$\vdots$$

$$r_k(k+1, t; n) < \mathrm{twr}_{t-1}(n^c)$$

Lower bounds

Erdős-Rado argument: $r_k(k+1, t; n) \leq 2^{(r_{k-1}(k, t-1; n))^{k-1}}$.

Theorem

$$c' \frac{n^{k-1}}{\log n} < r_k(k+1, 2; n) < c \frac{n^{k-1}}{\log n}$$

$$r_k(k+1, 3; n) < 2^{n^c}$$

$$r_k(k+1, 4; n) < 2^{2^{n^c}}$$

$$\vdots$$

$$r_k(k+1, t; n) < \mathrm{twr}_{t-1}(n^c)$$

Lower bounds

Erdős-Rado argument: $r_k(k+1, t; n) \leq 2^{(r_{k-1}(k, t-1; n))^{k-1}}$.

Theorem

$$c' \frac{n^{k-1}}{\log n} < r_k(k+1, 2; n) < c \frac{n^{k-1}}{\log n}$$

$$2^{c'n} < r_k(k+1, 3; n) < 2^{n^c}$$

$$r_k(k+1, 4; n) < 2^{2^{n^c}}$$

$$\vdots$$

$$r_k(k+1, t; n) < \mathrm{twr}_{t-1}(n^c)$$

Lower bounds

Erdős-Rado argument: $r_k(k+1, t; n) \leq 2^{(r_{k-1}(k, t-1; n))^{k-1}}$.

Theorem

$$c' \frac{n^{k-1}}{\log n} < r_k(k+1, 2; n) < c \frac{n^{k-1}}{\log n}$$

$$2^{c' n} < r_k(k+1, 3; n) < 2^{n^c}$$

$$2^{c' n} < r_k(k+1, 4; n) < 2^{2^{n^c}}$$

$$\vdots$$

$$r_k(k+1, t; n) < \mathrm{twr}_{t-1}(n^c)$$

Lower bounds

Erdős-Rado argument: $r_k(k+1, t; n) \leq 2^{(r_{k-1}(k, t-1; n))^{k-1}}$.

Theorem

$$c' \frac{n^{k-1}}{\log n} < r_k(k+1, 2; n) < c \frac{n^{k-1}}{\log n}$$

$$2^{c'n} < r_k(k+1, 3; n) < 2^{n^c}$$

$$2^{c'n} < r_k(k+1, 4; n) < 2^{2^{n^c}}$$

$$\vdots$$

$$2^{c'n} < r_k(k+1, t; n) < \mathrm{twr}_{t-1}(n^c)$$

New bounds for $r_k(k+1, t; n)$

Improvement on the Erdős-Rado upper bound argument.

Theorem (Mubayi-S. 2018)

For $k \geq 3$, and $2 \leq t \leq k$, we have

$$r_k(k+1, t; n) < \mathrm{twr}_{t-1}(cn^{k-t+1} \log n)$$

For $3 \leq t \leq k-2$

Theorem (Mubayi-S. 2018)

For $k \geq 6$, and $3 \leq t \leq k-2$, we have

$$\mathrm{twr}_{t-1}(c' n^{k-t+1}) < r_k(k+1, t; n) < \mathrm{twr}_{t-1}(cn^{k-t+1} \log n)$$

when $k - t$ is even, and

New bounds for $r_k(k+1, t; n)$

For $3 \leq t \leq k-2$

Theorem (Mubayi-S. 2018)

For $k \geq 6$, and $3 \leq t \leq k-2$, we have

$$\mathrm{twr}_{t-1}(c' n^{k-t+1}) < r_k(k+1, t; n) < \mathrm{twr}_{t-1}(c n^{k-t+1} \log n)$$

when $k - t$ is even, and

$$\mathrm{twr}_{t-1}(c' n^{(k-t+1)/2}) < r_k(k+1, t; n) < \mathrm{twr}_{t-1}(c n^{k-t+1} \log n)$$

when $k - t$ is odd.

New bounds for $r_k(k+1, t; n)$

For $t = k-1, k, k+1$

Theorem (Mubayi-S. 2018)

For $k \geq 6$,

$$\mathrm{twr}_{k-3}(cn^3) < r_k(k+1, k-1; n) < \mathrm{twr}_{k-2}(c'n^2 \log n)$$

$$\mathrm{twr}_{k-3}(cn^3) < r_k(k+1, k; n) < \mathrm{twr}_{k-1}(c'n \log n)$$

$$\mathrm{twr}_{k-2}(n^{c \log n}) < r_k(k+1, k+1; n) < \mathrm{twr}_{k-1}(c'n^2 \log n).$$

Open problem

$$2^{c' n^3} < r_5(6, 5; n) < 2^{2^{2^{n^c}}}$$

Improve the upper and lower bounds for $r_5(6, 5; n)$.

Thank you!