Infinite-valued Gödel Logics

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Takeuti and Titani [4] investigated the first-order version of infinite valued Gödel logic with the real interval $[0, 1]$ as its set of truth values. This logic was shown to be axiomatizable, but no cut-free system is know. We present such a cut-free axiomatization based on Avron's [1] hypersequent calculus for infinite-valued propositional Gödel logic. The proof of completeness of the system uses only the fact that the set of truth values contains a densely ordered subset (we call such sets *staturated*). It is thus shown that all Gödel logics based on such sets of truth values are the same. We show furthermore that any other infinite-valued first-order Gödel logic is not axiomatizable. (This was shown for a special case in [2].) In contrast to the propositional case, the order structure and not just the cardinality of the set of truth values determines the set of valid first-order formulas. The situation is analogous to the propositional case insofar as all Gödel logics based on saturated sets of truth values determine the same consequence relation, which is compact. All other infinite-valued Gödel logics are not compact [3].

References

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