On Global Closure Ordinals

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The *closure ordinal* $\mathrm{cl}(\varphi, \mathbf{A})$ of a first-order formula $\varphi(\bar{x}, X)$ on a structure $\mathbf{A}$ is the least ordinal ξ such that $\phi^{\xi}_{\mathbf{A}} = \phi^{<\xi}_{\mathbf{A}}$, where $\varphi^{\xi}_{\mathbf{A}} = \varphi^{<\xi}_{\mathbf{A}} \cup \{\bar{a} \mid \mathbf{A} \models \varphi(\bar{a}, \varphi^{<\xi}_{\mathbf{A}})\}$ and $\varphi^{<\xi}_{\mathbf{A}} = \bigcup_{\eta<\xi} \varphi^{\eta}_{\mathbf{A}}$. This is a classical *local* notion that has been extensively studied during the 1960s and the 1970s in the context of inductive definability on fixed infinite structures. Since the early 1980s, researchers in finite model theory have carried out a detailed study of *global* inductive definability on classes of finite structures, and have shown it to be intimately connected to polynomial-time computability. By and large, however, the study of global inductive definability on classes of infinite structures has remained unexplored.

We introduce and study the concept of the *global closure ordinal* $\mathrm{gcl}(\varphi)$ of a first-order formula $\varphi(\bar{x}, X)$, where $\mathrm{gcl}(\varphi)$ is the supremum of the closure ordinals $\mathrm{cl}(\varphi, \mathbf{A})$ over all structures $\mathbf{A}$. We establish that universal first-order formulas and positive existential-universal first-order formulas can realize every global closure ordinal, a result that sharply contrasts with the state of affairs in local inductive definability. We also characterize the class of global closure ordinals in terms of a certain Π^1_1-describability criterion, a result that makes it possible to derive further structural properties of global closure ordinals.

After this, we investigate the uniform closure ordinals of inductively definable queries, where the *uniform closure ordinal* of a query Q is the the infimum over the global closure ordinals of all possible inductive definitions of Q. Our main result here is that a query Q has uniform closure ordinal less than or equal to ω if and only if it is inductively definable by an existential first-order formula. Finally, using results from the model theory of infinitary logic and admissible fragments, we show that a countable admissible ordinal is a global closure ordinal if and only if it is the uniform closure ordinal of a query inductively definable by a universal first-order formula.