Countable structures with a fixed group of automorphisms

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Let $\mathcal{L}$ be a language which contains a countable infinity of unary function symbols or a relation or function symbol of arity at least 2. Let $X_{\mathcal{L}}$ be the Polish space of stuctures for $\mathcal{L}$ with universe $\mathbb{N}$, as in [2, 16.C]. We have the following descriptive set theoretic result:

Theorem 1. *Let G and H be countable groups. Then $\{x \in X_{\mathcal{L}} \mid \mathrm{Aut}(\mathcal{A}_x) \simeq G\}$ and $\{x \in X_{\mathcal{L}} \mid \mathrm{Aut}(\mathcal{A}_x) \simeq H\}$ are $\mathbf{\Pi}^1_1$-complete sets which are Borel inseparable.*

From the correspondence between $\mathcal{L}_{\omega_1\omega}$-sentences and invariant Borel subsets of $X_{\mathcal{L}}$ we get the following model theoretic interpretation of the preceding theorem:

Theorem 2. *Let G and H be countable groups and let σ be a sentence of $\mathcal{L}_{\omega_1\omega}$ satisfied by all countable structures for $\mathcal{L}$ whose automorphism group is isomorphic to G. Then there is a countable structure whose automorphism group is isomorphic to H satisfying σ.*

The result concerning complexity of $\{x \in X_{\mathcal{L}} \mid \mathrm{Aut}(\mathcal{A}_x) \simeq G\}$ given in theorem 1 cannot be generalised to uncountable groups. In fact we have:

Theorem 3. *Let p be a prime number. Then $\{x \in X_{\mathcal{L}} \mid \mathrm{Aut}(\mathcal{A}_x) \simeq \mathbb{Z}_p^{\mathbb{N}}\}$ is $\mathbf{\Pi}^1_1\&\mathbf{\Sigma}^1_1$-complete.*

Every countable structure can be coded in a countable graph. From that construction (see [1]) we obtain:

Theorem 4. *Let $L = \{R\}$ be the language consisting of one binary relation symbol and let $\mathcal{G} \subseteq X_L$ be the closed set of graphs. Then the results of theorems 1, 2 and 3 hold replacing everywhere $X_{\mathcal{L}}$ with $\mathcal{G}$.*

References

[1] W. Hodges: "Model theory", Cambridge University Press 1993.

[2] A.S. Kechris: "Classical descriptive set theory", Springer-Verlag 1995.

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