Section 6.6

(3/20/08)

Estimating definite integrals

In this section we discuss techniques for finding approximate values of definite integrals and work with applications where the data is given approximately by graphs and tables. We also present the Trapezoid and Simpson's Rules for approximating integrals, discuss upper and lower Riemann sums, and give error estimates for the Midpoint, Trapezoid, and Simpson's Rules.

Topics:

- ***Finding approximate integrals from graphs and tables***
- ***The Trapezoid Rule***
- ***Upper and lower Riemann sums***
- ***Simpson's Rule***
- ***Error estimates***

Finding approximate integrals from graphs and tables

The St. Francis dam, constructed in 1928 northeast of the present Magic Mountain near Los Angeles, was designed by William Mulholland with the plans he had used for the Mulholland dam that still supports the Hollywood Reservoir. The sides of the canyon where the St. Francis dam was built had geological flaws that were not recognized at the time. When the reservoir was filled for the first time, the dam broke, flooding the San Francisquito and Santa Clara River valleys and drowning 450 people. This tragedy ended the previously glamorous career of the self-educated engineer Mulholland, who had been the chief architect of the Los Angeles–Owens River Aqueduct that supplies much of the water to Los Angeles.

Example 1 Figure 1 shows the graph of the rate of flow of water $r = r(t)$ from the St. Francis dam in a 90-minute period starting 15 minutes before it broke.[1] Estimate the total volume of water that flowed from the dam for $0 \le t \le 90$.

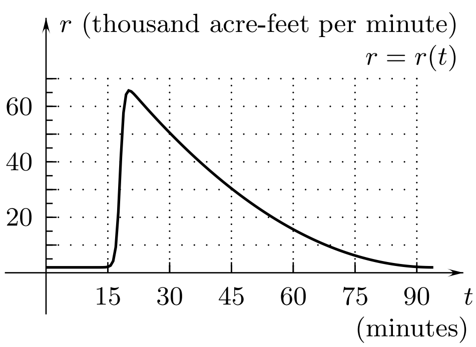


FIGURE 1

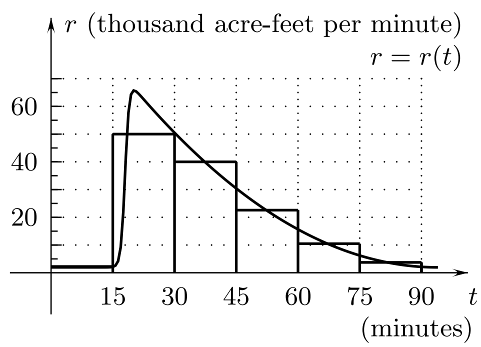


FIGURE 2

SOLUTION Suppose that $V(t)$ is the volume of water that has flowed from the dam from time 0 to time $t \ge 0$. Then $V'(t) = r(t)$, and since $V(0) = 0$, Part I of the Fundamental Theorem in Section 6.3 shows that the volume of water to flow from the dam for $0 \le t \le 90$ is

$$V(90) = V(90) - V(0) = \int_0^{90} r(t)\, dt.$$

Since $r = r(t)$ is a positive function, this integral is equal to the area of the region between its graph and the t-axis for $0 \le t \le 90$ in Figure 1.

[1] Data adapted from "A man, a dam and a disaster: Mulholland and the St. Francis Dam" by J. Rogers, *Ventura County Historical Society Quarterly*, Vol. 77, Ventura California: Ventura County Historical Society, 1995, p. 76.

We estimate this area by six rectangles whose sides are determined by the vertical lines at $t = 0, 15, 30, 45, 60, 75,$ and 90. We approximate each portion of the curved region by a rectangle, as in Figure 2, with the tops chosen to have the area of each rectangle appear approximately equal to the area of the corresponding portion of the region under the curve. The width of each rectangle is 15. From the sketch we estimate the heights of the rectangles to be 2, 50, 40, 23, 11, and 4, so that

$$\begin{aligned}[\text{Total volume}] &\approx (2)(15) + (50)(15) + (40)(15) + (23)(15)\\ &\quad + (11)(15) + (4)(15) = 1950 \text{ thousand acre-feet.}\end{aligned}$$

The areas of the rectangles are measured in units of thousand acre-feet, because the heights are measured in thousand acre-feet per minute and the widths in minutes. (An acre-foot is the volume of an acre of water one foot deep.) □

In the next example we use Riemann sums to estimate an integral of a function whose values are given in a table.

Example 2 The table below lists the rate $r = r(t)$ at which residents of the U.S. spent money on commodities and services, as measured on January 1 every other year just before and during the Great Depression.[2] **(a)** Express the total spent from the beginning of 1929 to the beginning of 1939 as an integral. **(b)** Estimate the integral by using the left Riemann sum corresponding to the partition $1929 < 1931 < 1933 < 1935 < 1937 < 1939$. **(c)** Estimate the integral by using the right Riemann sum.

Table 1. Rate of spending (billion dollars per year)

t	1929	1931	1933	1935	1937	1939
$r(t)$	77.2	60.5	45.8	55.7	66.5	72.0

[2] Data adapted from *Historical Statistics of the United States*, Washington, DC: U.S. Department of Commerce, p. 319.

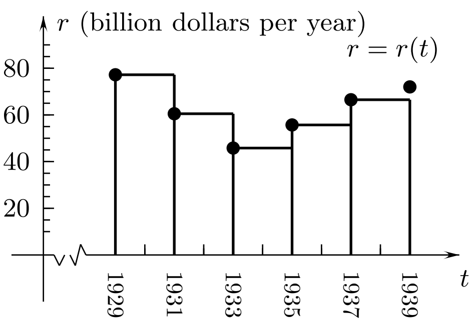
SOLUTION **(a)** The total spent from the beginning of 1929 to the beginning of 1939 is given by the integral $\int_{1929}^{1939} r(t)\,dt$ of the rate of spending.

(b) The values of r in the table are represented by the dots in Figure 3. We don't know any other of its values, but its graph would be a curve through those points. Figure 3 also shows the rectangles whose total area equals the left Riemann sum for the integral from part (a). Since the rectangles are all two units wide and their heights are 77.2, 60.5, 45.8, 55.7, and 66.5, the left Riemann sum is

$$(77.2)(2) + (60.5)(2) + (45.8)(2) + (55.7)(2) + (66.5)(2) = 611.4 \text{ billion dollars.}$$

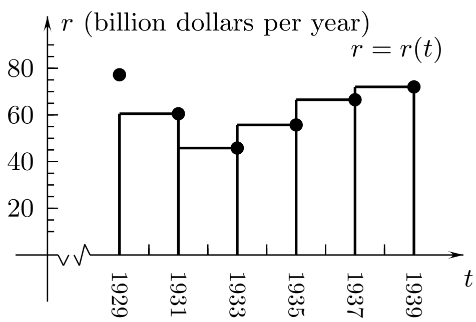
(c) The right Riemann sum is equal to the total area of the five rectangles in Figure 4 and is

$$(60.5)(2) + (45.8)(2) + (55.7)(2) + (66.5)(2) + (72.0)(2) = 601 \text{ billion dollars.} \ \square$$



Left Riemann sum

FIGURE 3



Right Riemann sum

FIGURE 4

The trapezoid rule

The left and right Riemann sums that correspond to the step functions in Figures 3 and 4 do not give very accurate estimates of the actual amount that was spent in the United States during the ten years from 1929 through 1938 because the rate of spending $r = r(t)$ was could not have been constant during any of the two-year periods in the decade. The rate of spending probably changed gradually from the beginning of one year to the end of the next.

We would expect to obtain a better estimate if we use a midpoint Riemann sum, but the values of the function at the midpoints, 1930, 1932, 1934, 1936, and 1938, are not given in the table. Instead we use the integral of the piecewise linear function $y = r_T(t)$ whose graph is in Figure 5. This graph is formed by line segments connecting the dots determined by the values of $r = r(t)$ in the table. This technique of approximating an integral is called the TRAPEZOID RULE because in the case of a positive function as in Figure 5, the region between the graph of the function and the horizontal axis consists of trapezoids and its integral is given by their total area.

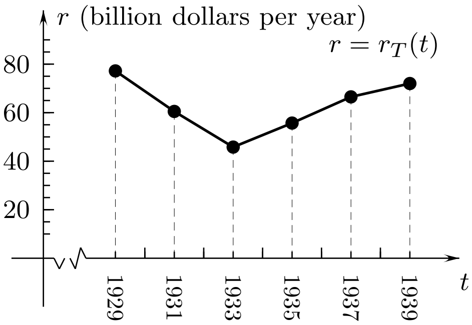


FIGURE 5

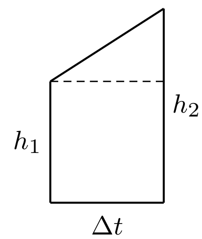


FIGURE 6

If the lengths of the vertical sides of a trapezoid are h_1 and h_2 with $h_2 \geq h_1$ and its width is Δt, as in Figure 6, then the trapezoid consists of a rectangle of area $h_1\Delta t$ and a triangle of area $\frac{1}{2}(h_2 - h_1)\Delta t$. Its area is therefore $h_1\Delta t + \frac{1}{2}(h_2 - h_1)\Delta t = \frac{1}{2}(h_2 + h_1)\Delta t$, which is the average of the lengths of the sides multipled by its width. Consequently, the Trapezoid Rule approximation for a positive function is the average of the left and right Riemann sums with the same partition. This is the also the case for functions that have negative or positive and negative values, so we are led to the following definition, which we state for a function $y = f(x)$.

Definition 1 (The Trapezoid Rule) *The Trapezoid Rule approximation of* $\int_a^b f(x)\,dx$ *corresponding to the partition* $a = x_0 < x_1 < x_2 < \cdots < x_N = b$ *of* $[a, b]$ *into* N *subintervals of equal width* Δx *is*

$$\sum_{j=1}^{N} \tfrac{1}{2}[f(x_{j-1}) + f(x_j)]\Delta x \tag{1a}$$

$$= \left[\tfrac{1}{2}f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{N-1}) + \tfrac{1}{2}f(x_N)\right]\Delta x. \tag{1b}$$

Formula **(1a)** expresses the trapezoid-rule approximation as the average of the left and right Riemann sums. Formula **(1b)** is obtained by combining terms in **(1a)** as follows:

$$\sum_{j=1}^{N} \tfrac{1}{2}[f(x_{j-1}) + f(x_j)]\Delta x$$

$$= \left[\tfrac{1}{2}[f(x_0) + f(x_1)] + \tfrac{1}{2}[f(x_1) + f(x_2)] + \cdots \tfrac{1}{2}[f(x_{N-2}) + f(x_{N-1})] + \tfrac{1}{2}[f(x_{N-1}) + f(x_N)]\right]\Delta x$$

$$= \left[\tfrac{1}{2}f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{N-1}) + \tfrac{1}{2}f(x_N)\right]\Delta x.$$

The fraction $\frac{1}{2}$ appears only with the $f(x_0)$ and $f(x_N)$ in the last expression because the other terms, $f(x_1), f(x_2), \ldots, f(x_{N-1})$, appear twice in the original sum.

Example 3 Use the Trapezoid Rule and the data in Table 1 to estimate the total amount that was spent in the U. S. on commodities and services from the beginning of 1929 to the beginning of 1939.

SOLUTION With the values Table 1 and formula **(1b)** with t in place of x, $\Delta t = 2$, $r(t_0) = 77.2, r(t_1) = 60.5, r(t_2) = 45.8, r(t_3) = 55.7, r(t_4) = 66.5$, and $r(t_5) = 72.0$ we obtain

$$[\text{Total spent for } 1929 \le t \le 1939] = \int_{1929}^{1939} r(t)\,dt$$
$$\approx [\tfrac{1}{2}(77.2) + 60.5 + 45.8 + 55.7 + 66.5 + \tfrac{1}{2}(72.0)](2) = 606.2 \text{ billion dollars.} \quad \square$$

Upper and lower Riemann sums

If $y = f(x)$ has a maximum value in each subinterval of the partition $a = x_0 < x_1 < \cdots < x_N = b$, then the UPPER RIEMANN SUM for the partition is

$$\int_a^b f(x)\,dx \approx \sum_{j=1}^{N} f(c_j)\Delta x_j = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_{N-1})\Delta x_{N-1} + f(c_N)\Delta x_N$$

where for each j, $f(c_j)$ is the greatest value of $f(x)$ in the jth subinterval. The upper Riemann sum is the greatest of all Riemann sums for $\int_a^b f(x)\,dx$ corresponding to that partition. We call it an UPPER APPROXIMATION or UPPER ESTIMATE of the integral because it is either greater than or equal to it.

We obtain the LOWER RIEMANN SUM by choosing $f(c_j)$ to be the least value of $f(x)$ in the jth subinterval for each j. The lower Riemann sum is the least of all Riemann sums for the partition. It is a LOWER APPROXIMATION or LOWER ESTIMATE of the integral.

Upper and lower Riemann sums are easiest to find if, as in the next example, the function is continuous and increasing or decreasing, since then the maxima and minima occur at endpoints of the subintervals.

Example 4 The following table gives the rate of oil production, measured in billion barrels per year, in Russia at the beginning of each year from 1990 through 1995.[3] The rate of production $r = r(t)$ decreased throughout this time period. Find upper and lower estimates of the total production for $1990 \le t \le 1995$.

TABLE 2. RATE OF OIL PRODUCTION IN RUSSIA
(BILLION BARRELS PER YEAR)

t	1990	1991	1992	1993	1994	1995
$r(t)$	3.7	3.3	2.8	2.5	2.2	2.1

[3] Data adapted from *Time*, May 27, 1996, p. 52, Source: PlanEcon.

SOLUTION The values of $r = r(t)$ from the table are represented by the dots in Figures 7 and 8. We assume that $r = r(t)$ is continuous so that its graph might look like the dashed curve in the drawings. The total production for $1990 \le t \le 1995$ is given by the integral $\int_{1990}^{1996} r(t)\,dt$ of the rate of production.

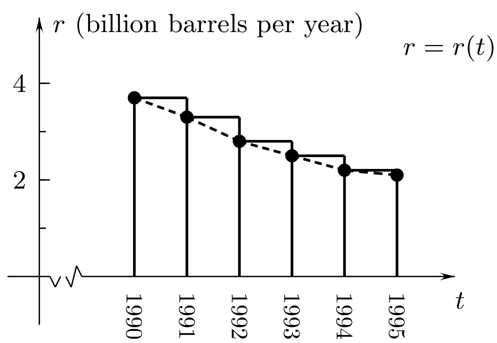
We can obtain upper and lower estimates of the total production by calculating upper and lower Riemann sums for the integral. Since $r = r(t)$ is a decreasing function its upper Riemann sum is the left Riemann sum, which equals the area of the five rectangles in Figure 7:

$$[\text{Upper Riemann sum}] = (3.7)(1) + (3.3)(1) + (2.8)(1) + (2.5)(1) + (2.2)(1) = 14.5.$$

The lower Riemann sum is the right Riemann sum, which equals the area of the five rectangles in Figure 8:

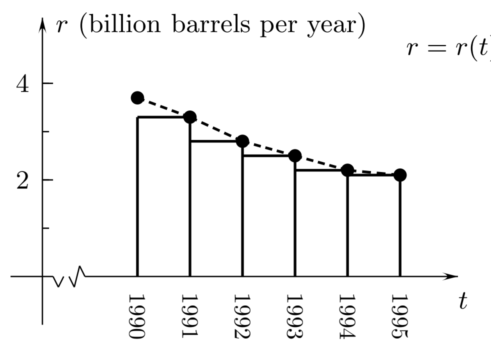
$$[\text{Lower Riemann sum}] = (3.3)(1) + (2.8)(1) + (2.5)(1) + (2.2)(1) + 2.1(1) = 12.9.$$

We conclude that, based on the data in Table 2, the total production of oil in Russia for $1990 \le t \le 1995$ was less than 14.5 billion barrels and greater than 12.9 million barrels. □



Upper Riemann sum

FIGURE 7



Lower Riemann sum

FIGURE 8

Simpson's Rule

If you are using a calculator or computer program to estimate a definite integral of a function given by an exact formula, you can usually achieve all the accuracy you require with Riemann sums or the Trapezoid Rule by using a sufficiently large number of subintervals. If, however, you have to use a fixed number of subintervals because the values of the function are given only in a table or for some other reason, you can generally obtain greater accuracy by using SIMPSON'S RULE,† with which portions of the graph are approximated by parabolas.

†Named after the English mathematician, Thomas Simpson (1710–1761).

To construct a Simpson-rule approximation of $\int_a^b f(x)\ dx$ for $a < b$, we use a partition $a = x_0 < x_1 < x_2 < \cdots < x_{2M} = b$ into an even number of subintervals of equal width, where M is a positive integer. We approximate the graph of $y = f(x)$ over the first two subintervals by the parabola through the points on the graph at $x = x_0, x = x_1$, and $x = x_2$; we approximate the graph over the next two subintervals by the parabola through the points at $x = x_2, x = x_3$, and $x = x_4$; and we continue in this manner until the entire graph has been approximated.

Figures 9 and 10 illustrate the case of $M = 3$, for which there are six subintervals. The values of a function $y = f(x)$ at the seven points $a = x_0, x_1, x_2, \ldots, x_6 = b$ are plotted in Figure 9. Figure 10 shows the three portions of parabolas that are used to approximate the entire graph of f.

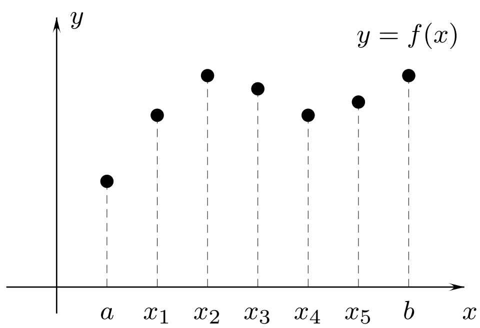


FIGURE 9

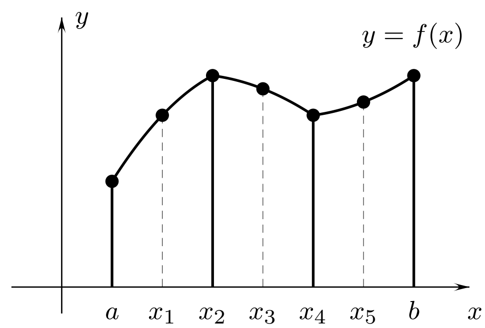


FIGURE 10

To obtain a formula for Simpson's Rule approximations in terms of values of the function f, we would need to find formulas for the quadratic polynomials whose graphs are used to approximate the graph of f and then integrate those formulas. We omit these calculations. The result is that for each k, the integral of the kth approximating quadratic polynomial from $x = x_{2k-2}$ to $x = x_{2k}$ is

$$\tfrac{1}{3}[f(x_{2k-2}) + 4f(x_{2k-}) + f(x_{2k})]\Delta x$$

where $\Delta x = (b-a)/(2M)$ is the width of the subintervals. Adding these expressions gives the following definition.

Definition 2 (Simpson's Rule) *The Simpson Rule approximation of* $\int_a^b f(x)\,dx$ *with* $2M$ *subintervals is*

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \sum_{k=1}^{M} \tfrac{1}{3}[f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})]\Delta x \\ &= \tfrac{1}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + 2f(x_6) + 4f(x_7) \\ &\qquad + \cdots + 2f(x_{2M-2}) + 4f(x_{2M-1}) + f(x_{2M})]\Delta x\end{aligned} \tag{2}$$

where $a = x_0 < x_1 < x_2 < \cdots < x_{2M-1} < x_{2M} = b$ *is the partition of* $[a, b]$ *into* $2M$ *subintervals of width* $\Delta x = (b-a)/(2M)$.

Notice that inside the square brackets on the right of **(2)**, all of the numbers $f(x_j)$ with an odd subscript j are multiplied by 4 and that, for even j, the numbers $f(x_j)$ except the first and the last ($f(x_0)$ and $f(x_{2M})$) are multiplied by 2.

Example 5 Find the Simpson rule approximation of $\int_0^1 \frac{1}{x^2+1}\,dx$ with four subintervals. Use six decimal places of accuracy.

SOLUTION We first calculate approximate decimal values of $f(x) = 1/(x^2+1)$ at the points in the partition $0 < 0.25 < 0.5 < 0.75 < 1$ of $[0,1]$:

x	0	0.25	0.5	0.75	1
$\frac{1}{x^2+1} \doteq$	1	0.941176	0.8	0.64	0.5

Since the width Δx of the subintervals is 0.25, the Simpson-Rule approximation is

$$\begin{aligned}\int_0^1 \frac{1}{x^2+1}\,dx &\approx \tfrac{1}{3}[f(0)+4f(0.25)+2f(0.5)+4f(0.75)+f(1)](0.25)\\ &\doteq \tfrac{1}{3}[1+4(0.941176)+2(0.8)+4(0.64)+0.5](0.25)\\ &\doteq 0.785392. \ \square\end{aligned}$$

Error estimates

A procedure for approximating the value of an integral is most useful if the error in the approximation can be estimated in advance, so we can determine in each instance how many subintervals to use to get the level of accuracy we need. Such error estimates for the Midpoint, Trapezoid and Simpson's Rules are given in the next theorem, which is established in advanced courses.

Theorem 1 (Error estimates with the Midpoint, Trapezoid, and Simpson's Rules)
(a) *If f has a continuous second derivative on $[a,b]$ and M_2 is the maximum of $|f''(x)|$ for $a \le x \le b$, then the Midpoint and Trapezoid Rule approximations of $\int_a^b f(x)\,dx$ with N equal subintervals of length $\Delta x = (b-a)/N$ satisfy*

$$\left|\int_a^b f(x)\,dx - \begin{bmatrix}\text{The Midpoint rule}\\ \text{approximation}\end{bmatrix}\right| \le \tfrac{1}{24}(b-a)M_2(\Delta x)^2 \tag{3}$$

$$\left|\int_a^b f(x)\,dx - \begin{bmatrix}\text{The Trapezoid rule}\\ \text{approximation}\end{bmatrix}\right| \le \tfrac{1}{12}(b-a)M_2(\Delta x)^2. \tag{4}$$

(b) *If f has a continuous fourth derivative on $[a,b]$ and M_4 is the maximum of $|f^{(iv)}(x)|$ for $a \le x \le b$, then the Simpson Rule approximation with N equal subintervals of length $\Delta x = (b-a)/N$ satisfies*

$$\left|\int_a^b f(x)\,dx - \begin{bmatrix}\text{The Simpson's rule}\\ \text{approximation}\end{bmatrix}\right| \le \tfrac{1}{180}(b-a)M_4(\Delta x)^4. \tag{5}$$

Example 6 **(a)** Calculate **(a)** the Midpoint, **(b)** the Trapezoid, and **(c)** the Simpson's Rule approximations of $\int_0^1 x^5 \, dx$ with four equal subintervals. **(d)** Use Theorem 1 to estimate the error made with the Midpoint Rule. **(e)** Find the exact value of the integral and use it to find the actual error with the Midpoint Rule.

SOLUTION **(a)** Because the interval $[0,1]$ of integration is 1 unit wide, the width of the subintervals in a partition into four equal subintervals is $\Delta x = \frac{1}{2}$ and the partition $0 = x_0 < x_1 < x_2 < x_3 < x_4 = 2$ is

$$0 < \tfrac{1}{4} < \tfrac{1}{2} < \tfrac{3}{4} < 1.$$

Since the midpoints of the subintervals are $c_1 = \frac{1}{8}, c_2 = \frac{3}{8}.c_3 = \frac{5}{8}$, and $c_4 = \frac{7}{8}$, the Midpoint Rule approximation is

$$\sum_{j=1}^{4} (c_j)^5 \, \Delta x = \left[\left(\tfrac{1}{8}\right)^5 + \left(\tfrac{3}{8}\right)^5 + \left(\tfrac{5}{8}\right)^5 + \left(\tfrac{7}{8}\right)^5\right]\left(\tfrac{1}{4}\right) \doteq 0.153930664. \qquad \textbf{(6)}$$

(b) The Trapezoid Rule approximation is

$$\begin{aligned}[\tfrac{1}{2}(x_0)^5 &+ (x_1)^5 + (x_2)^5 + (x_3)^5 + \tfrac{1}{2}(x_4)^5]\Delta x \\ &= \left[\tfrac{1}{2}(0^5) + (\tfrac{1}{4})^5 + (\tfrac{1}{2})^5 + (\tfrac{3}{4})^5 + \tfrac{1}{2}(1^5)\right]\left(\tfrac{1}{4}\right) = 0.26525.\end{aligned}$$

(c) Simpson's Rule gives

$$\begin{aligned}\tfrac{1}{3}[(x_0)^3 &+ 4(x_1)^3 + 2(x_2)^3 + 4(x_3)^3 + (x_4)^2)]\Delta x \\ &= \tfrac{1}{3}[0^3 + 4\left(\tfrac{1}{4}\right)^3 + 2\left(\tfrac{1}{2}\right)^3 + 4\left(\tfrac{3}{4}\right)^3 + 1^3]\left(\tfrac{1}{2}\right) = 0.16796875.\end{aligned}$$

(d) For $f(x) = x^5$, $f'(x) = 5x^4$, $f''(x) = 20x^3$, and the maximum of $|f''(x)|$ for $0 \le x \le 1$ is $M_2 = 20$. In this case $b - a = 1 - 0 = 1$ and $\Delta x = \frac{1}{4}$, so that estimate **(3)** gives

$$\begin{aligned}[\text{Error with the Midpoint Rule}] &\le \tfrac{1}{24}(b-a)M_2(\Delta x)^2 \\ &= \tfrac{1}{24}(1)(20)(\tfrac{1}{4})^2 \doteq 5.21 \times 10^{-2}. \qquad \textbf{(7)}\end{aligned}$$

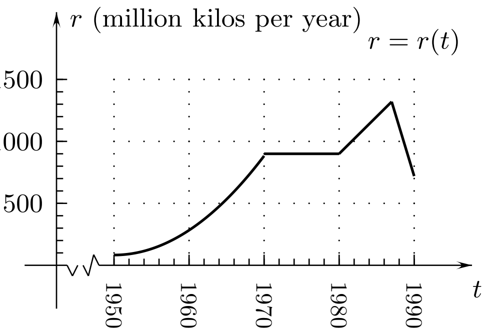
(e) Since the exact value of the integral is $\int_0^1 x^5 \, dx = \left[\frac{1}{6}\right]_{x=0}^{x=1} = \frac{1}{6}$, the error with the Midpoint Rule approximation **(5)** is $|\frac{1}{6} - 0.153930664| \doteq 1.2 \times 10^{-2}$, which is less than the error estimate **(7)**. □

Interactive Examples 6.6

Interactive solutions are on the web page http//www.math.ucsd.edu/~ashenk/.†

1. Chlorofluorocarbons (CFC's) that were used extensively in the past as coolants in refrigerators and air conditioners are catalysts in chemical processes that destroy ozone in the upper atmosphere. Figure 11 shows the graph of the rate of world production of CFC's from 1950 to 1990.[4] Estimate the total production of CFC's from the beginning of 1950 to the beginning of 1990.

FIGURE 11



2. The rate at which previously owned houses were being sold in the U. S. at the beginning of 1972, 1977, 1982, 1987, and 1992 is given in the table below.[5] Use the Trapezoid Rule to estimate the total number of sales for $1972 \le t \le 1992$.

Rate of sale
(million homes per year)

t	1972	1977	1982	1987	1992
$r(t)$	2.2	4.0	2.0	3.5	3.3

3. A car that slowing down has the velocities in the table below at times $t = 10, 12, 14, 16, 18$ and 20 (minutes). Use this data to give lower and upper estimates of how far it travels for for $10 \le t \le 20$.

t (minutes)	10	12	14	16	18	20
$v(t)$ (feet per minute)	90	85	75	60	40	0

4. Use **(a)** the Midpoint Rule, **(b)** the Trapezoid Rule, and **(c)** Simpson's Rule with four subintervals to estimate $\int_1^3 \sqrt{x}\, dx$. Find the exact value of the integral and use it to calculate the error that is made with each of the three approximations.

†In the published text the interactive solutions of these examples will be on an accompanying CD disk which can be run by any computer browser without using an internet connection.

(4)Data adapted from "Making environmental treaties work" by H. French, *Scientific American*, December, 1994, San Francisco: W. H. Freeman and Company, p. 97.

(5)Data adapted from *1993 Statistical Abstract of the United States*, Washington, DC: U. S. Department of Commerce, 1993, p. 710.

Exercises 6.6

[A]Answer provided. [O]Outline of solution provided. [C]Graphing calculator or computer required.

CONCEPTS:

1. Suppose that $y = f(x)$ is increasing for $a \le x \le b$. Explain why the midpoint Riemann sum for $\int_a^b f(x)\,dx$ with respect to any partition of $[a, b]$ is greater than the left Riemann sum and less than the right Riemann sum.

2. What can you say about a function $y = f(x)$ if is continuous on $[0, 4]$ and the upper and lower Riemann sums for $\int_0^4 f(x)\,dx$ relative to the partition $0 < 1 < 2 < 3 < 4$ are equal?

3. Use Theorem 1 to estimate the errors that are made with **(a)** the Trapezoid Rule and **(b)** Simpson's Rule in Example 6.

4. Use the error estimate **(5)** for Simpson's Rule to show that Simpson's Rule is exact for all integrals of polynomials of degree ≤ 3.

BASICS:

5.[O] Use three rectangles of equal width with the graph $y = f(x)$ in Figure 12 to find the approximate value of $\int_0^{15} f(x)\,dx$.

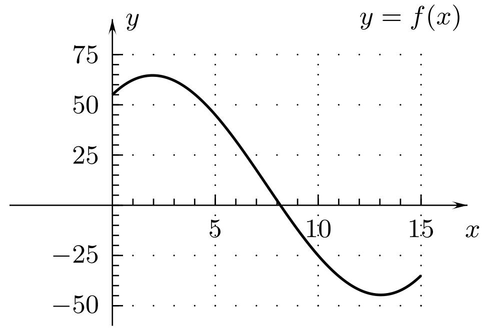


FIGURE 12

6.[A] Use an approximation by four rectangles of equal width to find the approximate value of $\int_0^4 F(x)\,dx$ for the function $y = F(x)$ of Figure 14.

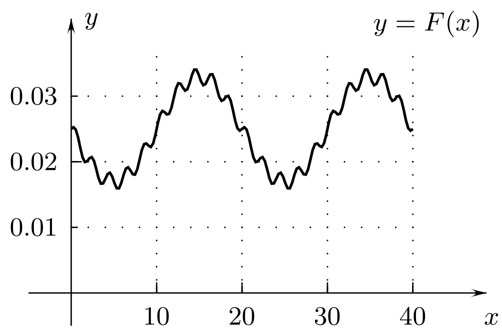


FIGURE 13

7. Use five rectangles of equal width to find the approximate value of $\int_0^5 H(x)\,dx$ for the function $y = H(x)$ of Figure 14.

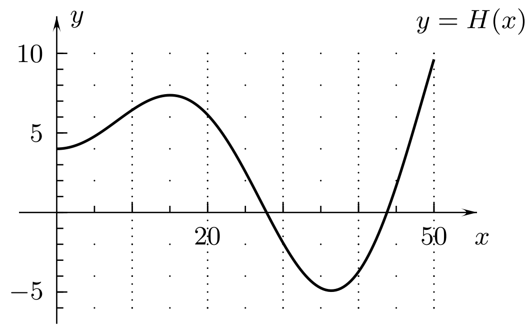


FIGURE 14

8.[O] Calculate the Trapezoid-Rule approximation of $\int_0^{15} g(x)\,dx$ with three subintervals for the function $y = g(x)$ of Figure 15. Then draw the trapezoids whose areas give the approximation.

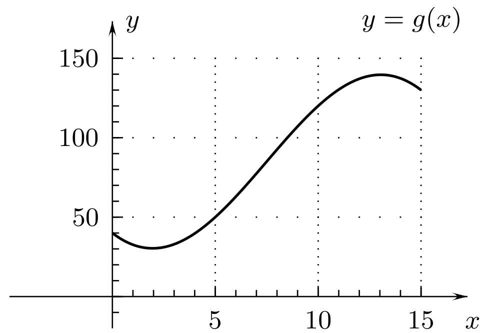


FIGURE 15

9. Use the Trapezoid Rule with three subintervalsto find the approximate value of $\int_0^3 J(x)\,dx$ for $y = J(x)$ in Figure 16.

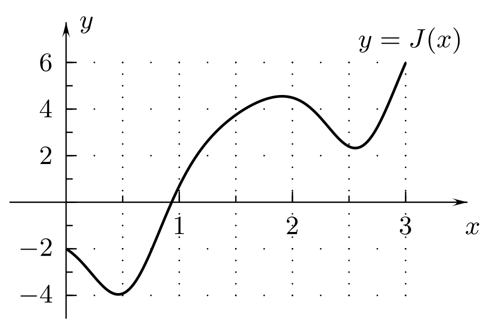


FIGURE 16

10.[O] Values of an increasing function $y = J(x)$ are given below. Calculate **(a)** the lower Riemann sum approximation and **(b)** the upper Riemann sum approximation of $\int_1^2 J(x)\,dx$ with five equal subintervals.

x	1.0	1.2	1.4	1.6	1.8	2.0
$J(x)$	30	32	33	38	40	45

11.[A] The graph of the rate of regeneration of a tadpole's lost tail is given in Figure 17. What is the approximate length of the tail after twenty days?

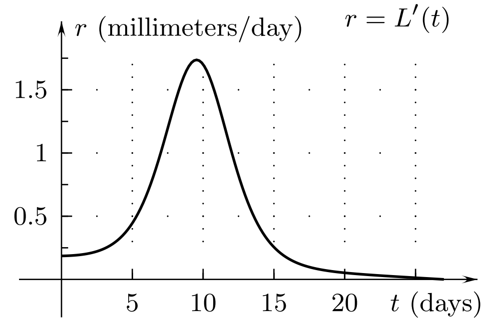


FIGURE 17

12. Figure 18 shows the graph of the rate $r = r(t)$ (milliliters per day) at which oxygen is taken into a kiwi egg during the 40 days between when the egg is laid and when it is hatched.[6] Approximately how much oxygen is taken during the entire time?

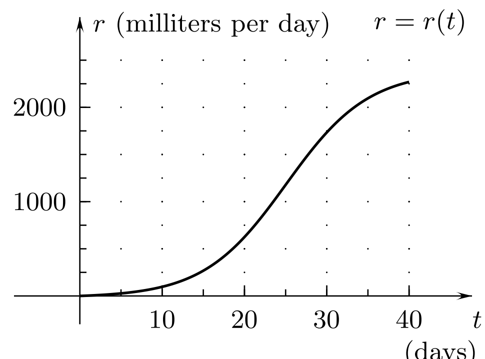


FIGURE 18

13.[A] The next table gives the rates at which three-piece steel and two-piece aluminum beverage cans were being produced in the U. S. at the beginnings of six years.[7] Use Trapezoid Rule approximations to estimate how many more aluminum cans were produced than steel cans during the 35 years from the beginning of 1965 to the beginning of 1990.

Rate of production (billion cans per year)

t	1965	1970	1975	1980	1985	1990
Steel	18	28	22	4	0	0
Aluminum	1	4	17	42	63	92

14. The next table gives the rate $r = r(t)$ (million metric tons per year) at which grain was produced in the world at ten-year intervals from 1950 to 1990.[8] Use the Trapezoid Rule to estimate the total world grain production from the beginning of 1950 to the beginning of 1990.

Year	1950	1960	1970	1980	1990
$r(t)$	631	847	1096	1447	1780

[6] Data adapted from "The Kiwi" by W. Calder III, *Scientific American*, July, 1978, New York: NY: Scientific American, Inc., p.140.

[7] Data adapted from "The aluminum beverage can" by W. Hosford and J. Duncan, *Scientific American*, Sept.,1994, New York, NY: Scientific American, Inc., p. 52.

[8] Data adapted from *Vital Signs*, 1992, p. 25. Source: USDA, *World Grain Database*.

EXPLORATION:

15.[O] Calculate approximate decimal values of **(a)** the Midpoint-Rule, **(b)** the Trapezoid-Rule, and **(c)** the Simpson-Rule approximations of $\int_0^1 x^4\, dx$ with four subintervals. **(d)** What are the errors in these approximations?

16.[O] Use **(a)** the Midpoint Rule, **(b)** the Trapezoid Rule, and **(c)** Simpson's Rule with four subintervals to estimate $\int_0^1 \cos x; dx$. **(d)** Find the exact value of the integral and calculate the errors in the three approximations. Use nine decimal places in the calculations.

17. Carry out the instructions of Problem 15 for $\int_0^1 e^x\, dx$ with four subintervals.

18.[O] Find **(a)** the upper Riemann sum and **(b)** the lower Riemann sum for $\int_0^2 (x^3 - 3x + 3)\, dx$ corresponding to the partition of $[0, 2]$ into four equal subintervals. Draw the rectangles whose areas give the sums.

19.[A] Calculate **(a)** the upper Riemann sum and **(b)** the lower Riemann sum for $\int_0^3 (8+3x-x^3)\, dx$ corresponding to the partition of $[0, 3]$ into three equal subintervals. Draw the rectangles whose areas give the sums.

20. Calculate **(a)** the upper Riemann sum and **(b)** the lower Riemann sum for $\int_{-1}^1 \frac{1}{x^2+1}\, dx$ corresponding to the partition of $[-1, 1]$ into four equal subintervals. Draw the rectangles whose areas give the sums.

21. Calculate with four-decimal-place accuracy, the upper and lower Riemann sums for $\int_1^3 (3 - \frac{2}{x^2})\, dx$ with four equal subintervals. Which is closer to the exact value of the integral?

22. As we will see in Section 6.8, the integral $\int_1^5 \frac{1}{x}\, dx$ equals $\ln(5)$, where $\ln x$ is the natural logarithm of x. Calculate the upper and lower Riemann sums for $\int_1^5 \frac{1}{x}\, dx$ corresponding to the partition into four equal subintervals. Which is closer to the exact value of the integral?

23.[A] A tank contains 1000 gallons of oil at $t = 0$ (hours). Figure 19 shows the graph of the rate of change of the volume for $0 \leq t \leq 50$. **(a)** When is oil flowing in and when is it flowing out of the tank? **(b)** Approximately how much oil is in the tank at $t = 50$? **(c)** What is the approximate maximum volume of oil in the tank?

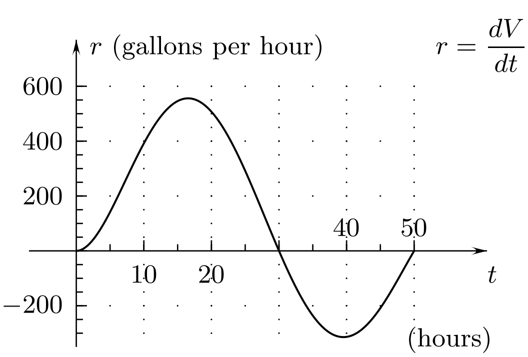


FIGURE 19

24. The function of Figure 20 gives the rate of oxygen intake by a man who bicycles strenuously for 20 minutes and then rests.[9] Approximately what percent of the oxygen that he breathes in for $0 \le t \le 40$ does he consume while bicycling?

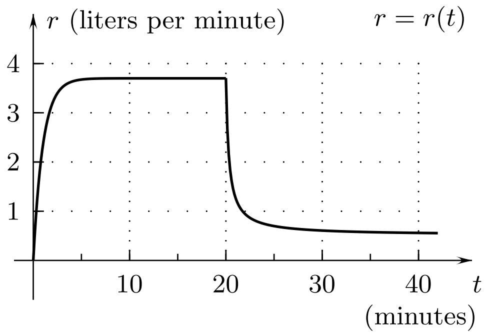


FIGURE 20

25.[A] The dots in Figure 21 give the rate at which world airlines were carrying passengers at the beginning of 1980, 1985, 1990, and 1995.[10] **(a)** Draw a line that approximates this data, and find an approximate formula for it. **(b)** The formula from part (a) gives a linear function $r = L(t)$ that approximates the rate of passenger use. Use it to give an estimate of the total number of airline passengers that were carried from the beginning of 1975 to the beginning of 1995.

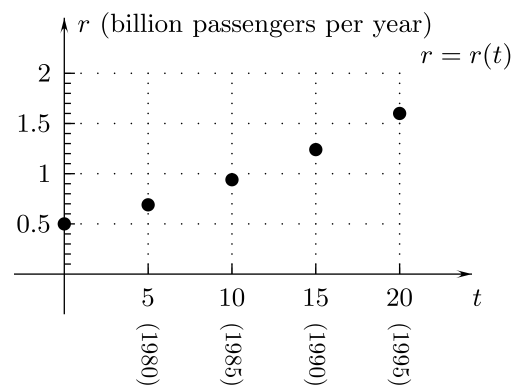


FIGURE 21

26. A golf club is swung at a golf ball at time $t = 0$ (seconds) and hits the ball at $t = 0.25$. The curve in Figure 22 is the graph of the rate of change $r = s'(t)$ of the distance that the head of the club travels up to time t.[11] Does the club head travel farther in the time interval $0 \le t \le 0.2$ or in the time interval $0.2 \le t \le 0.25$?

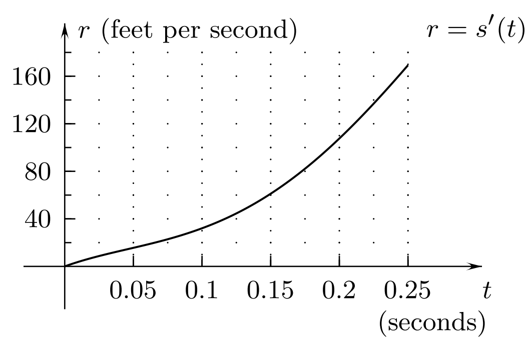


FIGURE 22

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[9] Data adapted from *Handbook of Physiology*, W. Fenn and H. Rahn editors, Washington, DC: American Physiological Society, 1965, p. 945.

[10] Data adapted from *New York Times*, December 9, 1996. New York, NY: The New York Times Company, Source: Airclaims Limited.

[11] Data adapted from *The Physics of Golf* by T. Jorgensen, Springer-Verlag, 1994, p. 15.

27.[A] A pool contains 100 gallons of water at $t = 1$ (hours), and the rate of flow of water into it is the linear function whose graph is the line in Figure 23. **(a)** Find an equation for the line. **(b)** Use the result of part (a) to find a formula for the volume of water $V = V(t)$ in the pool as a function of t.

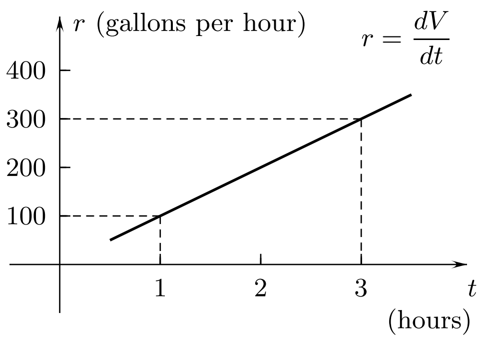

FIGURE 23

28. Figure 24 shows the upward acceleration of a space ship during the three stages of its launch.[12] The first and second parts of the curve correspond to the time periods when the space ship is being propelled by its first and second booster rockets, which are jettisoned. After that, the space ship is propelled by its own rocket. The acceleration is measured in g's (multiples of the acceleration due to gravity). **(a)** What is the approximate maximum acceleration of the space ship? **(b)** What is space ship's approximate upward velocity five minutes after take-off?

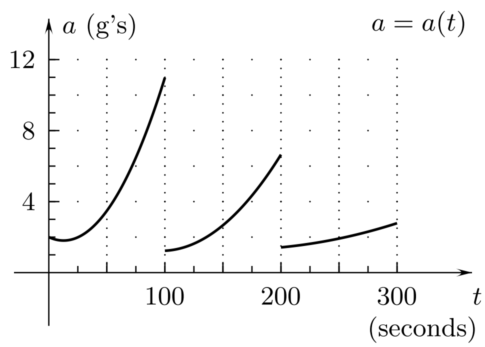

FIGURE 24

29. The graph of the rate of change of the temperature $T = T(t)^\circ$Fin a freezer as a function of time is given in Figure 25. The temperature is 25°F at $t = 1$ (minutes). **(a)** In what open subintervals of $1 \le t \le 16$ is the temperature increasing and in which is it decreasing? **(b)** Is the graph of the temperature concave up or concave down for $1 \le t \le 16$ and why? **(c)** What is the approximate temperature at $t = 16$?

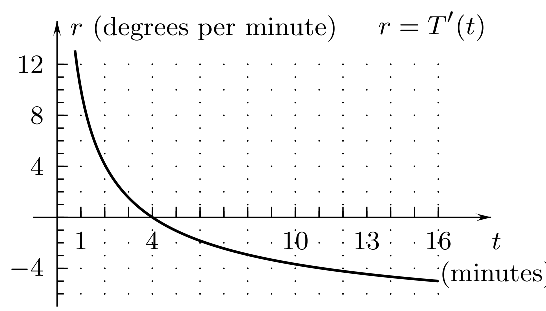

FIGURE 25

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[12] Data adapted from *Textbook of Medical Physiology* by A. Guyton, Philadelphia, PA: W. B..Saunders Co., 1981, p. 548.

30.[A] The table below lists the per-capita rate of consumption of wood $r(t)$ (cubic feet per person per year) and the population $P = P(t)$ (millions) in the United States at the beginnings of 1900, 1920, 1940, and 1960.[13] Based on this data, **(a)** what was the percent increase or decrease in the rate of consumption for the entire country from the beginning of 1900 to the beginning of 1960, and **(b)** approximately how much wood was consumed by the entire country from the beginning of 1900 to the beginning of 1960? (Use the Trapezoid Rule in part (b).)

Year	1900	1920	1940	1960
$r(t)$	157	113	85	63
$P(t)$	76	106	132	178

31. A 1793 epidemic of yellow fever killed approximately 10% of the population of Philadelphia, Pennsylvania. No accurate health records were kept, but a historian, Mattew Carey, estimated the death rate by counting new graves. His data is shown in Figure 26 with Carey's picture.[14] From this information he determined that approximately 4,000 died in the time period shown in the graph. What units are used on the vertical axis in the drawing? (The numbers on the vertical scale run from 0 to 120.)

FIGURE 26

32. Determine which is the greatest and which is the least of the three integrals $\int_2^5 (x^2+1)\,dx$, $\int_2^5 (x^3+x)\,dx$, and $\int_2^5 \frac{2}{x}\,dx$ without evaluating them. (Compare the integrands.)

C **33.** Use a Riemann-sum procedure on a calculator or computer to find a positive integer N such that the integral $\int_0^3 (5x^4 - 12x^2)\,dx$ differs form its Midpoint Rule approximation with N equal subintervals by less than 0.5 and more than 0.1.

34.[O] Calculate $\int_0^1 (x^4 + 2x)\,dx$ and its Midpoint Rule approximation with four equal subintervals. Then compare the error in the approximation with the estimate given by formula **(3)**.

35.[A] Calculate $\int_1^4 x^{3/2}\,dx$ and its Trapezoid Rule approximation with six equal subintervals. Then compare the error in the approximation with the estimate given by formula **(3)**.

[13] Data adapted from *Historical Statistics of the United States, Bicentennial Edition, Part 2*, Washington, DC: U. S. Bureau of the Census, 1975, p. 540.

[14] "The Philadelphia yellow fever epidemic of 1793" by K. Foster, M. Jenkins, and A. Toogood, *Scientific American*, August, 1988, New York, NY: Scientific American, Inc., p 93.

36. Calculate $\int_1^{0.8} x^{3/4}\,dx$ and its Midpoint Rule approximation with eight equal subintervals. Then compare the error in the approximation with the estimate given by formula **(2)**.

37. Calculate $\int_1^5 \sqrt{x}\,dx$ and its Simpson's Rule approximation with eight equal subintervals. Then compare the error in the approximation with the estimate obtained from Theorem 1.

(End of Section 6.6)