CHAPTER 14

(3/22/08)

Derivatives with Two or More Variables

Many mathematical models involve functions of two or more variables. The elevation of a point on a mountain, for example, is a function of two horizontal coordinates; the density of the earth at points in its interior is a function of three coordinates; the pressure in a gas-filled balloon is a function of its temperature and volume; and if a store sells a thousand items, its profit might be studied as a function of the amounts of each of the thousand items that it sells. This chapter deals with the differential calculus of such functions. We study functions of two variables in Sections 14.1 through 14.6. We discuss vertical cross sections of graphs in Section 14.1, horizontal cross sections and level curves in Section 14.2, partial derivatives in Section 14.3, Chain Rules in Section 14.4, directional derivatives and gradient vectors in Section 14.5, and tangent planes in Section 14.6. Functions with three variables are covered in Section 14.7 and functions with more than three variables in Section 14.8.

Section 14.1

Functions of two variables

OVERVIEW: *In this section we discuss domains, ranges and graphs of functions with two variables.*

Topics:

- ***The domain, range, and graph of*** $\mathbf{z = f(x, y)}$
- ***Fixing*** $\mathbf{x}$ ***or*** $\mathbf{y}$***: vertical cross sections of graphs***
- ***Drawing graphs of functions***

The domain, range, and graph of $\mathbf{z = f(x, y)}$

The definitions and notation used for functions with two variables are similar to those for one variable.

Definition 1 *A* FUNCTION f *of the two variables* x *and* y *is a rule that assigns a number* $f(x, y)$ *to each point* (x, y) *in a portion or all of the* xy*-plane.* $f(x, y)$ *is the* VALUE *of the function at* (x, y)*, and the set of points where the function is defined is called its* DOMAIN*. The* RANGE *of the function is the set of its values* $f(x, y)$ *for all* (x, y) *in its domain.*

If a function $z = f(x, y)$ is given by a formula, we assume that its domain consists of all points (x, y) for which the formula makes sense, unless a different domain is specified.

Example 1 **(a)** What is the domain of $f(x, y) = x^2 + y^2$? **(b)** What are the values $f(2, 3)$ and $f(-2, -3)$ of this function at $(2, 3)$ and $(-2, -3)$? **(c)** What is its range?

SOLUTION **(a)** Because the expression $x^2 + y^2$ is defined for all x and y, the domain of f is the entire xy-plane.
(b) $f(2, 3) = 2^2 + 3^2 = 13$ and $f(-2, -3) = (-2)^2 + (-3)^2 = 13$.

(c) The values $x^2 + y^2$ of the function are all nonnegative and for every $z \geq 0$ it has the value z at all points (x, y) on the circle $x^2 + y^2 = z$. Consequently, the range of f is the closed infinite interval $[0, \infty)$. □

Recall that the graph of a function f of one variable is the curve $y = f(x)$ in an xy-plane consisting of the points (x, y) with x in the domain of the function and $y = f(x)$. The graph of a function of two variables is a surface in three-dimensional space.

Definition 2 *The graph of a function f with the two variables x and y is the surface $z = f(x, y)$ formed by the points (x, y, z) in xyz-space with (x, y) in the domain of the function and $z = f(x, y)$.*

For a point (x, y) in the domain of the function, its value $f(x, y)$ at (x, y) is determined by moving vertically (parallel to the z-axis) from (x, y) in the xy-plane to the graph and then horizontally (parallel to the xy-plane) to $f(x, y)$ on the z-axis, as is shown in Figure 1.

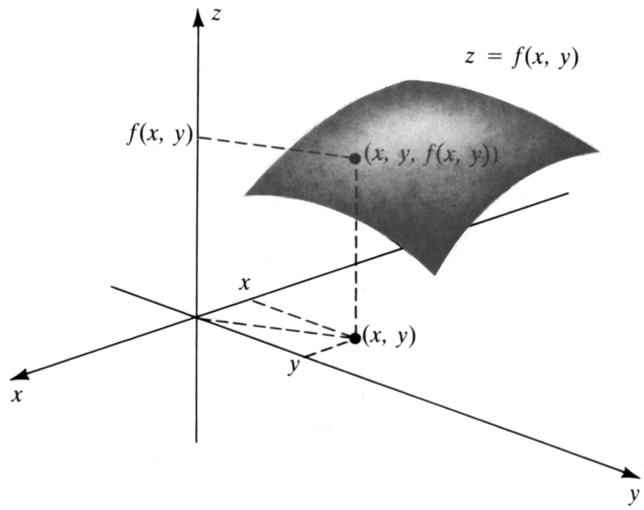


FIGURE 1

Fixing x *or* y*: vertical cross sections of graphs*

One way to study the graph $z = f(x, y)$ of a function of two variables is to study the graphs of the functions of one variable that are obtained by holding x or y constant. To understand this process, we need to look at the geometric significance of setting x or y equal to a constant.

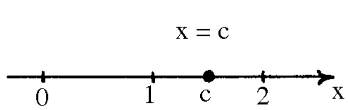
The meaning of the equation $x = c$, with c a constant, depends on the context in which it is used. If we are dealing with points on an x-axis, then the equation $x = c$ denotes the set[†]

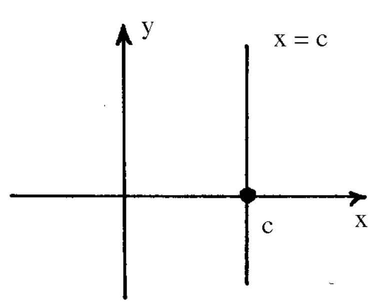
$$\{x : x = c\}$$

consisting of the one point with x-coordinate c (Figure 2). If we are talking about points in an xy-plane, then the equation $x = c$ denotes the vertical line

$$\{(x, y) : x = c\}$$

in Figure 3 consisting of all points (x, y) with $x = c$. Notice that this line is perpendicular to the x-axis and intersects it at $x = c$.





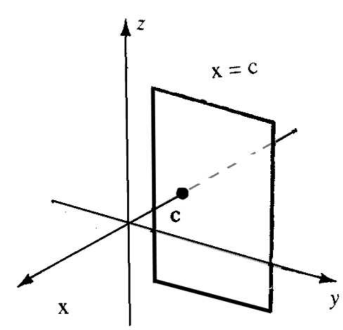


FIGURE 2 FIGURE 3 FIGURE 4

[†]We are using here SET-BUILDER NOTATION $\{P : \boldsymbol{Q}\}$ for the set of points P such that condition $\boldsymbol{Q}$ is satisfied.

When we are dealing with xyz-space, the equation $x = c$ denotes the vertical plane

$$\{(x, y, z) : x = c\}$$

consisting of all points (x, y, z) with $x = c$. This is the plane perpendicular to the x-axis (parallel to the yz-plane) that intersects the x-axis at $x = c$. With the axes oriented as in Figure 4, this plane is in front of the yz-plane if c is positive and behind the yz-plane if c is negative. (c is negative in Figure 4.)

Similarly, $y = c$ denotes the point $\{y : y = c\}$ on a y-axis (Figure 5), denotes the horizontal line $\{(x, y) : y = c\}$ perpendicular to the y-axis at $y = c$ in an xy-plane (Figure 6), and denotes the vertical plane $\{(x, y, z) : y = c\}$ perpendicular to the y-axis (parallel to the xz-plane) at $y = c$ in xyz-space. With axes oriented as in Figure 7, this plane is to the right of the xz-plane for positive c and to the left of the xz-plane for negative c. (c is positive in Figure 7.)

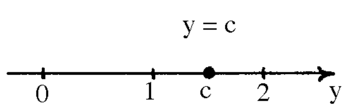


FIGURE 5

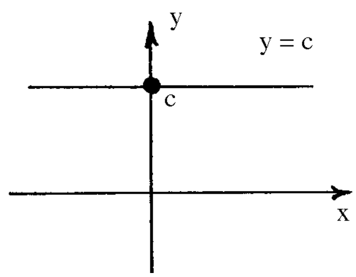


FIGURE 6

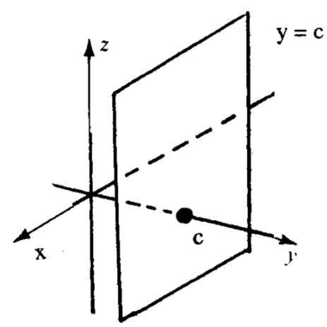


FIGURE 7

In the next examples the shapes of surfaces are determined by studying their cross sections in vertical planes $y = c$ and $x = c$.

Example 2 Determine the shape of the surface $z = x^2 + y^2$ **(a)** by studying its cross sections in the planes $x = c$ perpendicular to the x-axis and **(b)** by studying the cross sections in the planes $y = c$ perpendicular to the y-axis.

SOLUTION **(a)** The intersection of the surface $z = x^2 + y^2$ with the plane $x = c$ is determined by the simultaneous equations,

$$\begin{cases} z = x^2 + y^2 \\ x = c. \end{cases}$$

Replacing x by c in the first equation yields the equivalent pair of equations,

$$\begin{cases} z = c^2 + y^2 \\ x = c. \end{cases}$$

These equations show that the intersection is a parabola in the plane $x = c$ that opens upward and whose vertex is at the origin if $c = 0$ and is c^2 units above the xy-plane if $c \neq 0$. It has the shape of the curve $z = c^2 + y^2$ in the yz-plane of Figure 8.

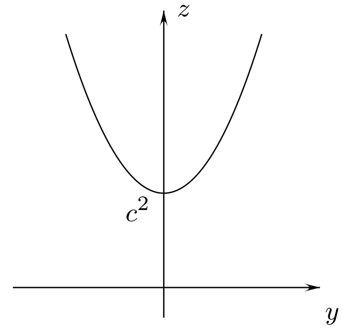


FIGURE 8

The curve drawn with a heavy line in the picture of xyz-space in Figure 9 is the cross section in the plane $x = c$ for one positive c. The other curves are the cross sections for other values of c. (In the drawings the cross sections are cut off at a positive value of z; they in fact, extend infinitely high.)

Notice how the cross sections change as c changes in Figure 9. For positive c, the plane $x = c$ is in front of the yz-plane and the vertex (lowest point) of the cross section is above the xy-plane. As c decreases to 0, the plane moves back toward the yz-plane and the vertex moves to the origin. Then as c decreases through negative values, the plane $x = c$ moves further toward the back and the vertex rises from the xy-plane. This gives the surface the bowl shape shown in Figure 10.

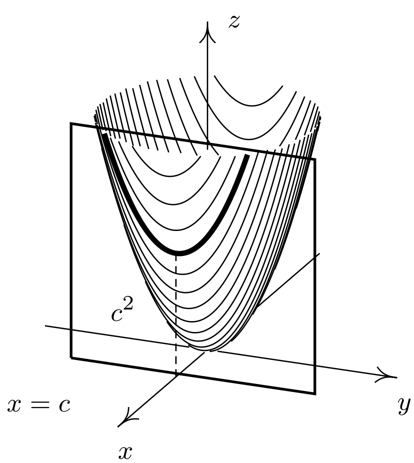


FIGURE 9

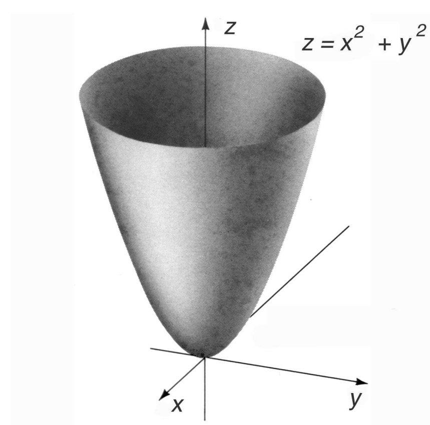


FIGURE 10

(b) The intersection of the surface $z = x^2 + y^2$ with the plane $y = c$ is determined by the simultaneous equations,

$$\begin{cases} z = x^2 + y^2 \\ y = c. \end{cases}$$

We replace y by c in the first equation to obtain the equivalent equations,

$$\begin{cases} z = x^2 + c^2 \\ y = c. \end{cases}$$

These show that the intersection is a parabola in the plane $y = c$ that opens upward and whose vertex is at the origin if $c = 0$ and is c^2 units above the xy-plane if $c \neq 0$. The curve for one positive c is the heavy curve in Figure 11. As c decreases toward 0, the plane $y = c$ moves to the left toward the xz-plane and the vertex moves down to the origin. Then as c decreases farther through negative values, the plane moves farther to the left and rises above the xy-plane. This gives the surface the bowl shape in Figure 10. (Figure 12 shows the cross sections from Figures 10 and 11 together.) □

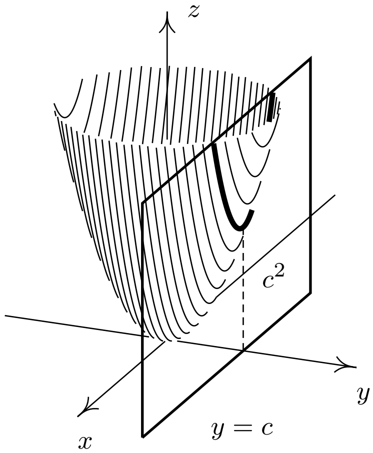


FIGURE 11

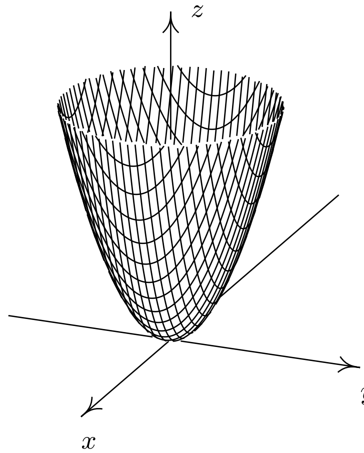


FIGURE 12

The surface in Figure 12 is called a CIRCULAR PARABOLOID because its vertical cross sections are parabolas and, as we will see in the next section, its horizontal cross sections are circles.

Example 3 Determine the shape of the surface $z = y^2 - x^2$ **(a)** by studying its cross sections in the planes $x = c$ perpendicular to the x-axis and **(b)** by studyng the cross sections in the planes $y = c$ perpendicular to the y-axis.

SOLUTION **(a)** We follow the procedure from Example 2. The intersection of the surface $z = y^2 - x^2$ with the plane $x = c$ is determined by the simultaneous equations,

$$\begin{cases} z = y^2 - x^2 \\ x = c \end{cases}$$

which are equivalent to

$$\begin{cases} z = y^2 - c^2 \\ x = c. \end{cases}$$

The intersection is a parabola in the plane $x = c$ that opens upward and whose vertex is at $z = -c^2$, as shown in Figure 13 for a positive value of c. As c decreases to 0, the plane moves toward the yz-plane and the vertex rises to the origin. Then as c decreases through negative values, the plane $x = c$ moves toward the back and the vertex drops below the xy-plane. This gives the surface the saddle shape shown in Figure 14.

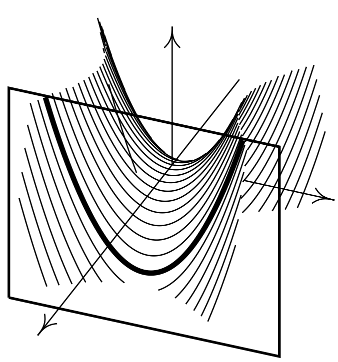


FIGURE 13

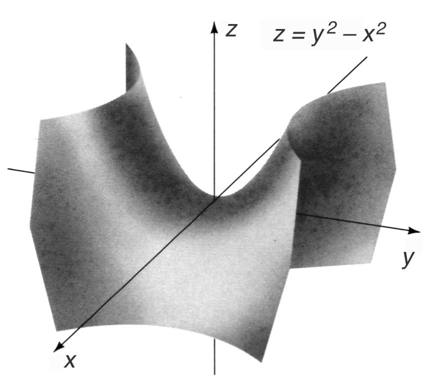


FIGURE 14

(b) The intersection of the surface $z = y^2 - x^2$ with the plane $y = c$ is determined by the equations,

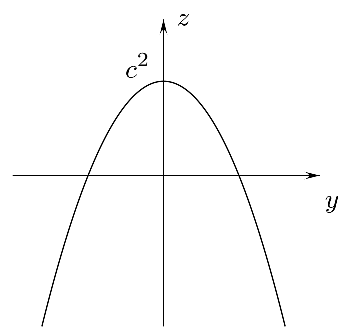
$$\begin{cases} z = y^2 - x^2 \\ y = c \end{cases}$$

which are equivalent to

$$\begin{cases} z = c^2 - x^2 \\ y = c. \end{cases}$$

The intersection is a parabola in the plane $y = c$ that opens downward and whose vertex is at the origin if $c = 0$, and is c^2 units above the xy-plane if $c \neq 0$ (Figure 15). The curve for one positive c is the heavy curve in Figure 16. As c decreases toward 0, the plane $y = c$ moves to the left toward the xz-plane and the vertex moves down to the origin. Then as c decreases further through negative values, the plane moves farther to the left and the vertex rises above the xy-plane. This again gives the surface the saddle shape in Figure 16. Figure 17 shows the two types of cross sections together. □

FIGURE 15



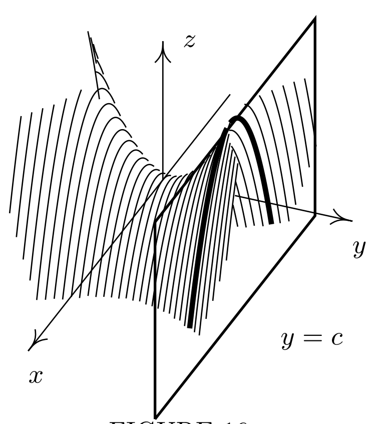


FIGURE 16

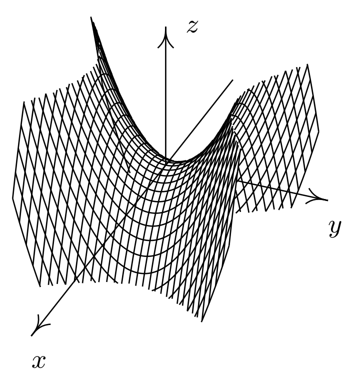


FIGURE 17

The surface in Figure 17 is called a HYPERBLIC PARABOLOID because its vertical cross sections are parabolas and, as we will see in the next section, its horizontal cross sections are hyperbolas.

Example 4 Determine the shape of the surface $z = y - \frac{1}{12}y^3 - \frac{1}{4}x^2$ by analyzing its cross sections in the planes $x = c$, perpendicular to the x-axis.

SOLUTION The cross section of the surface in the vertical plane $x = c$ has the equations

$$\begin{cases} z = y - \frac{1}{12}y^3 - \frac{1}{4}x^2 \\ x = c \end{cases}$$

which are equivalent to

$$\begin{cases} z = y - \frac{1}{12}y^3 - \frac{1}{4}c^2 \\ x = c. \end{cases} \qquad \textbf{(1)}$$

We first consider the case of $c = 0$, where the cross section **(1)** is the curve $z = y - \frac{1}{12}y^3$ in the yz-plane. The derivative $z' = \dfrac{d}{dy}(y - \frac{1}{12}y^3) = 1 - \frac{1}{4}y^2 = \frac{1}{4}(4 - y^2)$ is zero at $y = \pm 2$, negative for $y < -2$ and $y > 2$, and positive for $-2 < y < 2$. Consequently, this curve has the shape shown in Figure 18.

The cross section **(1)** with $x = c$ with nonzero c has the shape of the curve in Figure 18 lowered c^2 units. As c increases from 0 through positive values the cross section moves down and in the positive x-direction and, and as c decreases from 0 through negative values it moves down and in the negative x-direction. This gives the surface the boot-like shape in Figure 19. □

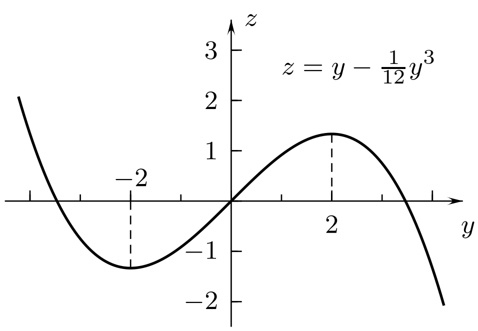


FIGURE 18

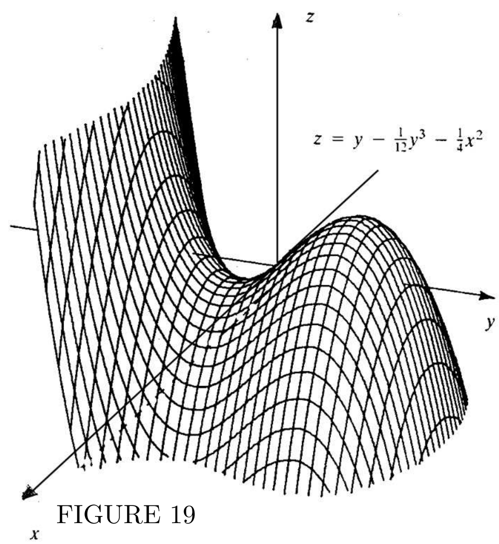


FIGURE 19

We could also determine the shape of the surface in Figure 19 by studying the cross sections of the surface in the plane $y = c$ perpendicular to the y-axis. They have the equations,

$$\begin{cases} z = y - \frac{1}{12}y^3 - \frac{1}{4}x^2 \\ y = c \end{cases}$$

which are equivalent to

$$\begin{cases} z = c - \frac{1}{12}c^3 - \frac{1}{4}x^2 \\ y = c. \end{cases} \qquad \textbf{(2)}$$

For $c = 0$, this cross section is is the parabola $z = -\frac{1}{4}x^2$ in the xy-plane of Figure 20. It passes through the origin and opens downward.

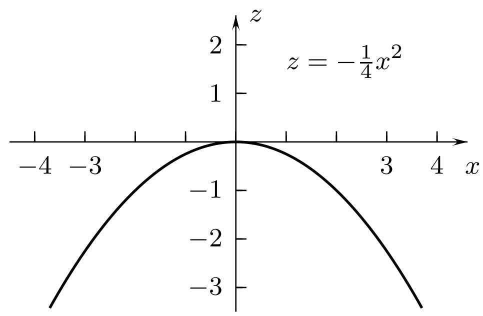


FIGURE 20

The cross section **(2)** in the plane $y = c$ has the same parabolic shape for nonzero c. It is to the right of the origin if c is positive and to the left of the origin if c is negative. The height of its vertex is the value of $c - \frac{1}{12}c^3$. Hence, the cross section moves up and to the right as c increases from 0 to 2 and moves down and to the right as c increases beyond 2. It moves down and to the left as c decreases from 0 to -2 and moves up and to the left as c decreases beyond -2. This gives the surface the shape in Figure 19.

Another way to visualize the surface in Figure 19 is to imagine that wire hoops with the shape of the parabola in Figure 19 are hung in perpendicular planes on the curve in Figure 17.

Example 5 Determine the shape of the surface $h(x, y) = -\frac{1}{8}y^3$.

SOLUTION The graph $z = -\frac{1}{8}y^3$ is especially easy to analyze because the equation does not involve the variable x. This implies that if one point (x_0, y_0, z_0) with y-coordinate y_0 and z-coordinate z_0 is on it, then the line parallel to the x-axis formed by the points (x, y_0, z_0) as x ranges over all numbers is on the surface. Consequently, the surface consists of lines parallel to the x-axis. In this case, the intersection of the surface with the yz-plane, where $x = 0$, is the curve $z = -\frac{1}{8}y^3$, and the surface in Figure 21 consists of this curve and all lines through it parallel to the x-axis. □

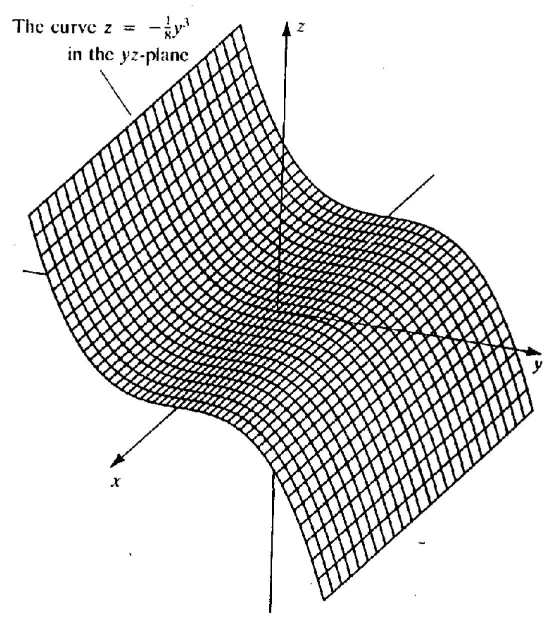


FIGURE 21

We will find in Chapter 15 that the surfaces $z = kxy$ with nonzero constants k are important in the study of maxima and minima of functions with two variables. We cannot determine the shapes of the graphs of these functions from their cross sections in planes $y = c$ and $x = c$ since those intersections are lines while the surfaces are curved. These surfaces are best understood by rotating coordinates as in Figure 22, which shows shows the result of establishing x' and y'-axes by rotating the x- and y-axes 45° counterclockwise.† According to Theorem 3 of Section 12.2, the original coordinates (x, y) can be calculated from the new coordinates (x', y') by the formulas,

$$x = \frac{1}{\sqrt{2}}(x' - y'),\ y = \frac{1}{\sqrt{2}}(x' + y'). \tag{3}$$

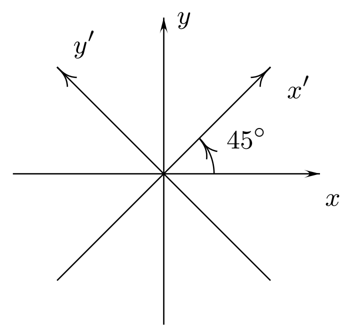


FIGURE 22

Example 6 Use $x'y'$-coordinates as in Figure 22 to analyze the surface $z = -2xy$.

SOLUTION Equations **(3)** transform the equation $z = -2xy$ into

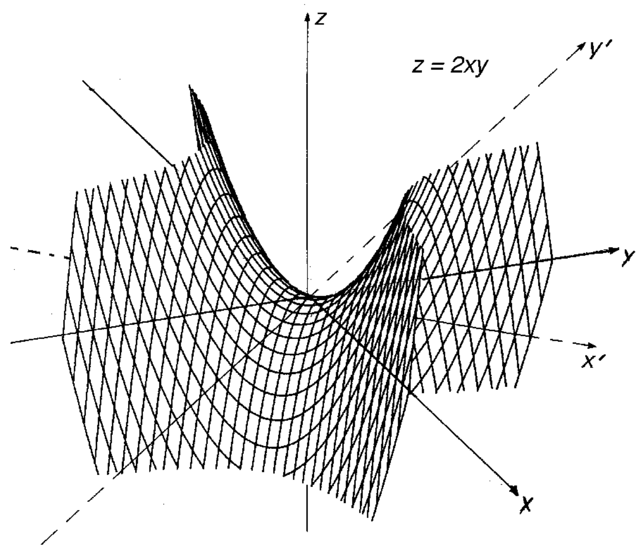
$$z = -2\left[\frac{1}{\sqrt{2}}(x' - y')\right]\left[\frac{1}{\sqrt{2}}(x' + y')\right].$$

This simplifies to $z = -(x' - y')(x' + y')$ and then to

$$z = (y')^2 - (x')^2. \tag{4}$$

This is the equation of Example 3 with x and y replaced by x' and y'. Consequently, the graph is the surface of Figure 17 rotated 45° as in Figure 23. Notice that the vertical cross sections of the graph in the planes $y' = c$ perpendicular to the y'-axis are parabolas that open upward, and the vertical cross sections in the planes $x' = c$ perpendicular to the x'-axis are parabolas that open downward, as can be seen from equation **(4)**. □

†The primes on the variables x' and y' here are just to distinguish them from x and y. They do not denote derivatives.



(To be redrawn)
FIGURE 23

Drawing graphs of functions

In earlier chapters when we were dealing with functions of one variable, we could rely extensively on hand-drawn sketches or graphs generated by calculators or computers to guide our reasoning and check our work. In dealing with functions of two variables, we generally have to reason more abstractly because it is difficult to draw good pictures of most surfaces, and drawings of surfaces generated by graphing calculators or computers are frequently difficult to interpret. Moreover, even when we have a good sketch of the graph of a function with two variables, we often cannot determine the function's values from it because we cannot tell from the two-dimensional picture where vertical lines intersect the graph.

You should, however, be able to sketch paraboloids and other surfaces given by simple equations that are easy to analyze. Draw a portion of the surface in an imaginary box with sides parallel to the coordinate planes. Draw any profiles of the surface that are inside the box and any curves that are obtained by chopping off the surface at the sides of the box. Then add coordinate axes to match your drawing. For example, to draw a circular or elliptic paraboloid $z = a + bx^2 + cx^2$ or $z = a - bx^2 - cx^2$ with $b > 0, c > 0$, as in Figure 24, first draw a horizontal circle to represent the circle or ellipse where the surface is chopped off. Add a parabola for the profile of the surface. Then draw coordinate axes as appropriate.

$z = a + bx^2 + cy^2$ $b > 0, c > 0$ $\quad$ $z = a - bx^2 - cy^2$ $b > 0, c > 0$

FIGURE 24

To draw a hyperbolic paraboloid $z = a + bx^2 - cy^2$ or $z = a - bx^2 + cy^2$ with $b > 0, c > 0$ or $z = kxy$ with $k \neq 0$, as in Figure 25, first draw the "saddle seat" of the surface with part of a parabola that opens upwards. Add portions of horizontal hyperbolas where the surface is chopped off at the top and bottom, and draw parabolas or vertical lines to represent where the surface is chopped off at the sides. Then draw coordinate axes as appropriate.

Many graphs of functions $f(x, y) = G(x)$ or $f(x, y) = G(y)$ that depend on only one of the variables x or y are easy to sketch because their cross sections in one direction are horizontal lines and in the other

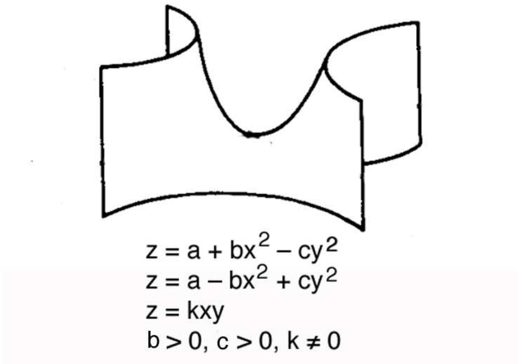


FIGURE 25

direction have the shape of the graph of the function G of one variable. For example, to draw the surface $z = a + bx^2, z = a + by^2, z = a - bx^2$, or $z = a - by^2$ with $b > 0$, as in Figure 26, draw parts of vertical parabolas to represent parabolic ends of the surface where it is cut off by vertical planes and draw three parallel lines to represent its profile and where is it chopped off by a horizontal plane.

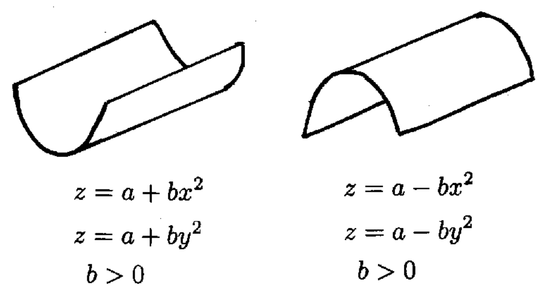


FIGURE 26

Interactive Examples 14.1

Interactive solutions are on the web page http//www.math.ucsd.edu/˜ashenk/.[†]

1. Add axes to the surface in Figure 27 so that it represents the surface $z = x^2 + y^2 + 1$.

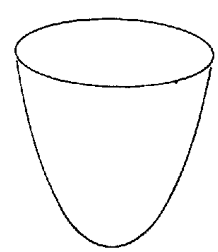


FIGURE 27

[†]In the published text the interactive solutions of these examples will be on an accompanying CD disk which can be run by any computer browser without using an internet connection.

2. Label the positive ends of the x- and y-axes in Figure 28 so that the surface is $z = y^2 - x^2$.

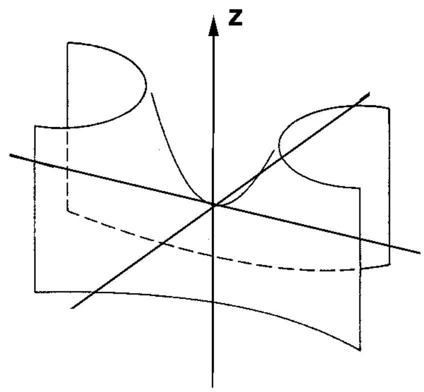


FIGURE 28

3. What is $f(1, -1)$ for $f(x, y) = x^2 e^{-y} - y^3 e^x$?

4. Draw the surface $z = -xy$.

5. Match the equations **(a)** $z = \sin y - \frac{1}{9}x^3$ and **(b)** $z = \sin y - \frac{1}{9}x^3 + \frac{1}{2}$ to their graphs in Figures 29 and 30 by studying their cross sections in the vertical planes $y = c$ perpendicular to the y-axis.

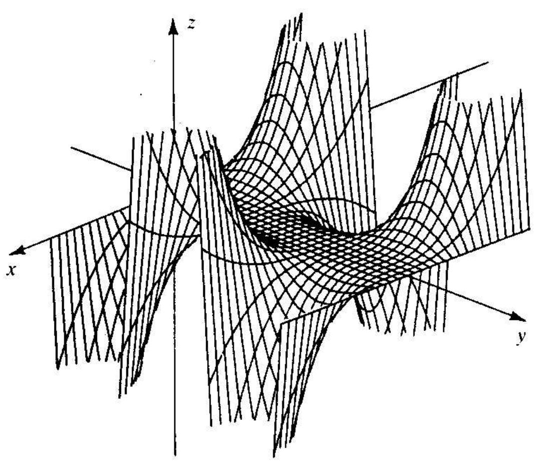


FIGURE 29

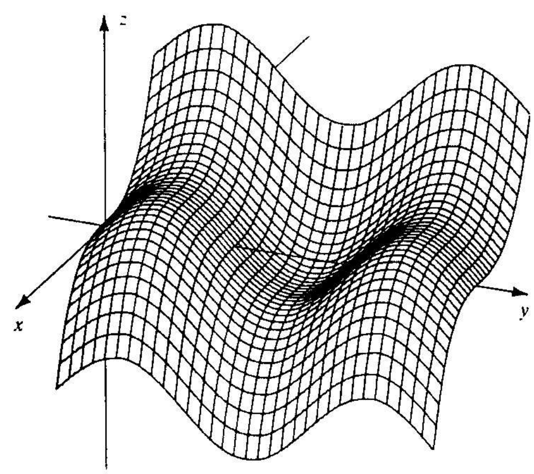


FIGURE 30

6. Match the equations **(a)** $z = \sin y - \frac{1}{9}x^3$ and **(b)** $z = \sin y - \frac{1}{9}x^3 + \frac{1}{2}$ to their graphs in Figures 29 and30 by studying their cross sections in the vertical planes $x = c$ perpendicular to the x-axis.

Exercises 14.1

AAnswer provided. **O**Outline of solution provided. **C**Graphing calculator or computer required.

CONCEPTS:

1. **(a)** Does increasing c move the plane $x = c$ toward the front or to the back if the axes are oriented as in Figure 4? **(b)** Does increasing c move the plane $y = c$ toward the left or to the right if the axes are oriented as in Figure 7?

2. How is the graph of $z = f(x, y)$ related **(a)** to the graph of $z = -f(x, y)$, **(b)** to the graph of $z = 2f(x, y)$, and **(c)** to the graph of $z = f(2x, 2y)$?

3. How is the graph of $z = f(x, y)$ related **(a)** to the graph of $z = f(x, y) + 1$, **(b)** to the graph of $z = f(x + 1, y)$, and **(c)** to the graph of $z = f(x, y + 1)$?

4. How is the graph of $z = f(x, y)$ related to the graph of $z = f(y, x)$?

BASICS:

Find the indicated values of the functions in Exercises 5 through 8.

5.[O] Calculate $f(2, 3)$ and $f(-1, 10)$ for $f(x, y) = x^2y^3$.

6.[A] $g(1, 3)$ for $g(x, y) = \dfrac{x - y}{x + y}$

7.[A] $z = \ln(xy + 3)$ at $x = 4, y = 5$

8. $h(3, 2)$ for $h(x, y) = x^y + y^x$?

9.[O] **(a)** Describe the intersections of the graph $z = y^2$ with the planes $x = c$ and $y = c$ for constants c. **(b)** Copy the surface in Figure 31, add axes, and label them so the drawing represents the graph from part (a).

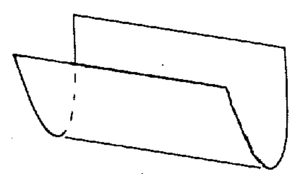


FIGURE 31

10.[O] **(a)** Figure 32 shows the surface $z = -\sqrt{1 - x^2 - y^2}$ Explain why it is the lower half of the sphere $x^2 + y^2 + z^2 = 1$ of radius 1 with its center at the origin. **(b)** Draw the graph of the function $g(x, y) = 2 - \sqrt{1 - x^2 - y^2}$.

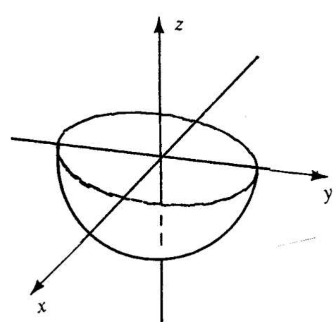


FIGURE 32

11.[A] Label the positive ends of the x- and y-axes in Figure 33 so that the surface is the graph of $Q(x,y) = \sqrt{1-y^2}$.

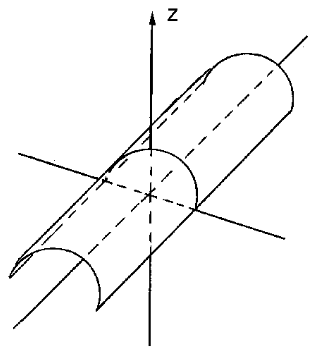


FIGURE 33

12. Label the positive ends of the x- and y-axes in Figure 34 so that the surface has the equation $z = |x|$.

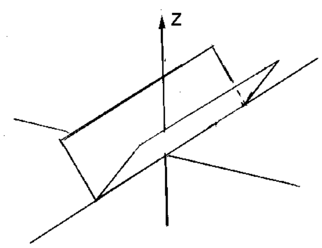


FIGURE 34

13.[O] Draw the graph of the function $L(x,y) = -3$.

14. Draw the graph of $V(x,y) = -\frac{1}{2}\sqrt{x^2+y^2}$.

15.[A] **(a)** What is the domain of $z(x,y) = \dfrac{x}{\sqrt{y}}$? **(b)** For what values of x and y is the function from part (a) positive?

16.[O] **(a)** Is $g(x,y) = x^3y$ an increasing or decreasing function of x for $y = 10$? **(b)** Is $g(x,y) = x^3y$ an increasing or decreasing function of y for $x = -2$?

17.[O] What is the global minimum of $h(x,y) = (x-y)^2 + 10$ and at what values of x and y does it occur?

18. What is the global maximum of $k(x,y) = \dfrac{6}{x^2+y^2+2}$ and at what point (x,y) does it occur?

19.[A] Use a calculator to determine which is the greatest and which is the least of the numbers $M(1,2)$, $M(2,1)$, and $M(2,2)$ for $M(x,y) = \dfrac{1+\cos x}{2+\sin y}$.

20.[A] The total resistance $R = R(r_1, r_2)$ of an electrical circuit consisting of resistances of r_1 and r_2 ohms (Figure 35) is determined by the equation $\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$.[1] Does $R(r_1, r_2)$ increase or decrease as r_1 or r_2 increases?

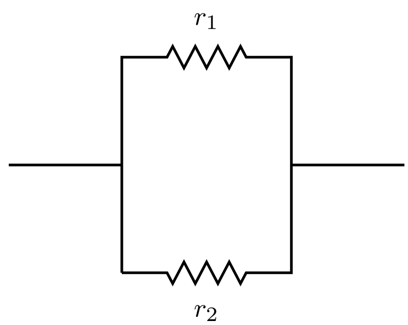


FIGURE 35

21. When a very small spherical pebble falls under the force of gravity in a deep body of still water, it quickly approaches a constant speed, called its TERMINAL SPEED. By STOKES' LAW the terminal speed is $v(r, \rho) = 21800(\rho - 1)r^2$ centimeters per second if the radius of the pebble is r centimeters and its density is $\rho \geq 1$ grams per cubic centimeter.[2] **(a)** What is the terminal speed of a quartz pebble of density 2.6 grams per cubic centimeter if its radius is 0.01 centimeters? **(b)** Which of two pebbles has the greater terminal speed if they have the same density and one is larger than the other? **(c)** What happens to the pebble if its density is 1 gram per centimeter, the density of water?

EXPLORATION:

22. The table below gives the equivalent human age $A = A(t, w)$ of a dog that is t years old and weighs w pounds.[3] **(a[A])** What does $A(11, 50)$ represent and, based on the table, what is its approximate value? **(b)** What does $A(14, 70)$ represent and what is its approximate value?

$A(t, w) =$ EQUIVALENT HUMAN AGE

	$t = 6$	$t = 8$	$t = 10$	$t = 12$	$t = 14$	$t = 16$
$w = 20$	40	48	56	64	72	80
$w = 50$	42	51	60	69	78	87
$w = 90$	45	55	66	77	88	99

(1) *CRC Handbook of Chemistry and Physics*, Boca Raton FL: CRC Press, Inc., 1981, p. F-112.
(2) *CRC Handbook of Chemistry and Physics*, Boca Raton FL: CRC Press, Inc., 1981, p. F-115.
(3) Data from "Senior-Care Health Report" by Pfizer, Inc., based on a chart developed by F. Menger, State College, PA.

In Exercises 23 through 26 match the functions to their graphs in Figures 36 through 39 and explain how the shapes of the surfaces are determined by their equations.

23.[O] $z = -\frac{1}{9}x^3 \sin y$

24.[A] $z = \sin y$

25. $z = -\sin x \sin y$

26. $z = 3e^{-x/5} \sin y$

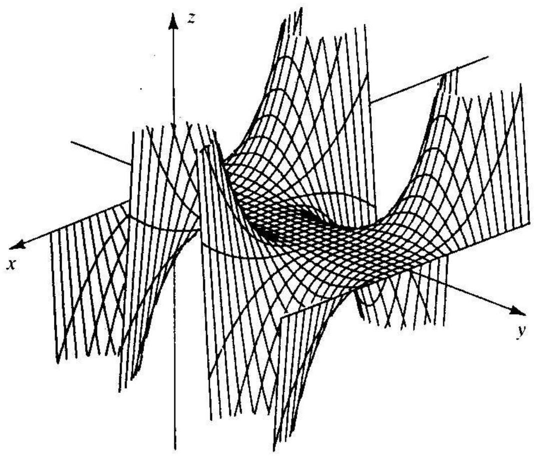


FIGURE 36

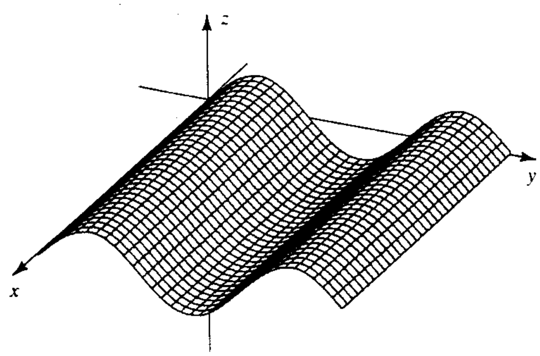


FIGURE 37

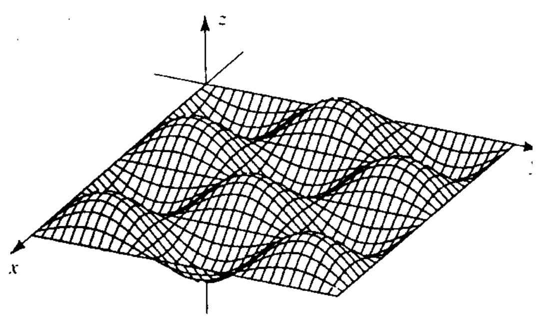


FIGURE 38

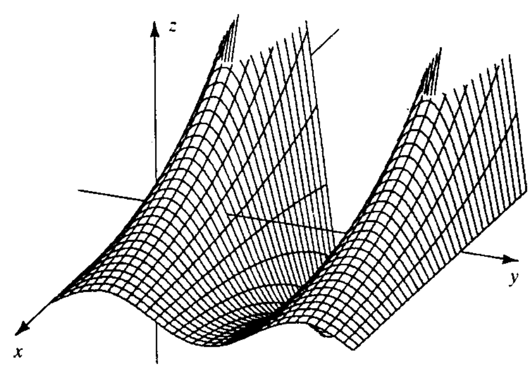


FIGURE 39

27. A person's BODY-MASS INDEX is the number $I(w,h) = \dfrac{w}{h^2}$, where w is his or her weight, measured in kilograms, and h is his or her height, measured in meters. **(a)** What is your body-mass index? (A kilogram is 2.2 pounds and a meter is 39.37 inches.) **(b)** A study of middle-aged men found that those with a body-mass index of over 29 had twice the risk of death than those whose body-mass index was less than 19. Suppose a man is 1.5 meters tall and has a body-mass index of 29. How much weight would he have to lose to reduce his body-mass index to 19?

28. Sketch the graph of $H(x,y) = \begin{cases} x^2 + y^2 & \text{for} \quad x^2 + y^2 < 1 \\ 2 - (x^2 + y^2) & \text{for} \quad x^2 + y^2 \geq 1 \end{cases}$.

29. **(a)** What is the domain of $z = \dfrac{1}{\sqrt{1 - xy}}$? **(b)** What is its global maximum value and at what points does it occur?

30. **(a)** What is the domain of $\sin^{-1}(x^2+y^2-2)$? **(b)** What are its global maximum and minimum values and at what points do they occur?

31.[A] Figure 40 shows the graph of $g(x,y) = \dfrac{10\cos(xy)}{1+2y^2}$. Find, without using derivatives, the global maximum of $z = g(x,y)$ and the points (x,y) where it occurs.

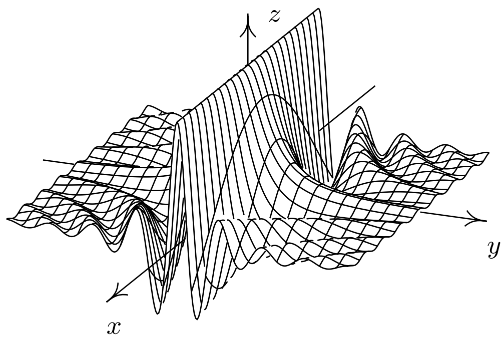


FIGURE 40

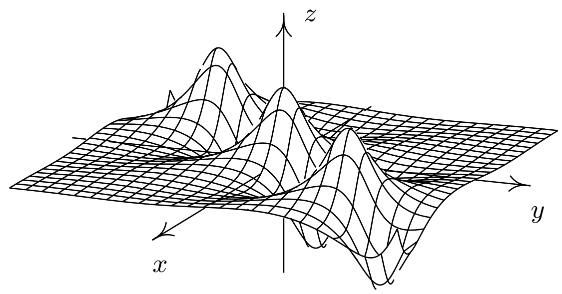


FIGURE 41

32. Find, without using derivatives, the global maximum of $h(x,y) = \dfrac{3\cos(x+y)}{1+(x-y)^2}$. and the values of (x,y) where they occur. The graph of $z = h(x,y)$ is in Figure 41.

33. Match the functions **(a$^{\mathsf{A}}$)** $A(x,y) = 2\cos(x+y) - x$, **(b)** $B(x,y) = 2\sin(y^2) + \dfrac{8}{1+x^2+y^2}$, **(c)** $C(x,y) = 4\cos(x+y) + \frac{1}{3}(x^2+y^2)$, and **(d)** $D(x,y) = 2\sin(2\pi y) + x\sin(\pi y)$ to their graphs in Figures 42 through 45 and explain in each case how the shape of the surface is determined by the formula for the function.

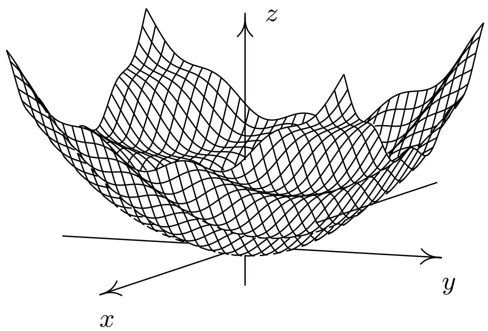


FIGURE 42

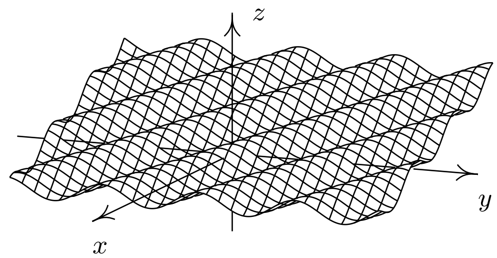


FIGURE 43

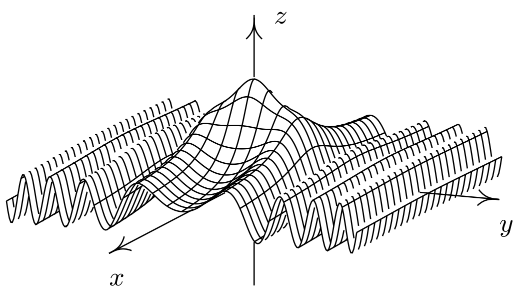


FIGURE 43

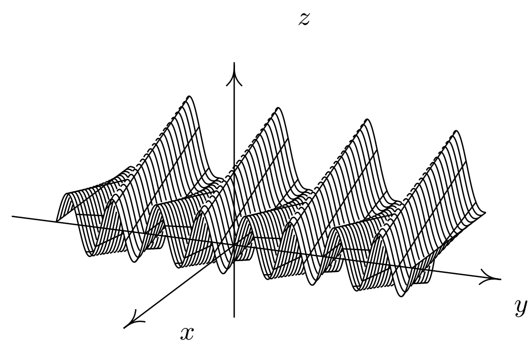


FIGURE 45

34. Describe and draw the graph of $G(x,y) = -1 - (x+1)^2 - (y-1)^2$.

35. **(a)** What is the domain of $z = \dfrac{1}{\sqrt{1-xy}}$? **(b)** What is its global maximum value and at what points does it occur?

36. **(a)** What is the domain of $\sin^{-1}(x^2+y^2-2)$? **(b)** What are its global maximum and minmimum values and at what points do they occur?

37. Sketch the graph of $H(x,y) = \begin{cases} x^2+y^2 & \text{for} \quad x^2+y^2 < 1 \\ 2-(x^2+y^2) & \text{for} \quad x^2+y^2 \geq 1 \end{cases}$.

(End of Section 14.1)