Summary of lectures of Weeks 8, 9 and 10.

One of the extremely important applications of differentiation is in solving optimization problems.

For a given multivariable function, how can we find its maximum and minimum?

Def. (1) We say f has a local maximum at P_0 if $f(P_0) \geq f(P)$ for any P in an open disk centered at P_0.

(2) We say f has a local minimum at P_0 if $f(P_0) \leq f(P)$ for any P in an open disk centered at P_0.

We know that, if $\nabla f(P_0) \neq 0$, then, in the direction of $\nabla f(P_0)$, f increases, and, in the direction of $-\nabla f(P_0)$, f decreases. Hence, if, at P_0, f has either a local max or a local min, then

$$\nabla f(P_0) = 0.$$

Def. A point P_0 is called a critical point of f

if either $\nabla f(P_0)$ does not exist or $\nabla f(P_0) = \vec{0}$.

The above argument implies that

If f has either a local max or a local min at P_0, then P_0 is a critical point.

A critical point can be neither a local max nor a local min, in which case it is called a saddle point.

The second derivative test can help us determine if a critical point is a local max, a local min or a saddle point.

Critical pt	$D = f_{xx} \cdot f_{yy} - (f_{xy})^2$	f_{xx}	Result
P_0	$+$	$+$	local min.
	$+$	$-$	local max.
	$-$		Saddle pt.
	0		Inconclusive.

Here the assumption is that all the second partial derivatives exist and continuous.

<u>Exp</u>. Let $f(x,y) = x^2 - xy + y^2 - y$.

(a) Find the critical points of f.

(b) Determine if they are local max, local min or saddle points.

<u>Solution</u>. (a) $\nabla f = \langle 2x - y,\ 2y - x - 1 \rangle = \langle 0, 0 \rangle$

$$\left.\begin{array}{l} 2x - y = 0 \\ -x + 2y = 1 \end{array}\right\} \Rightarrow -x + 2(2x) = 1$$

$$\Rightarrow \quad x = \frac{1}{3}$$

$$\Rightarrow \quad y = \frac{2}{3}.$$

It has only one critical point $\left(\frac{1}{3}, \frac{2}{3}\right)$.

(b) By the 2nd derivative test, we have

Critical pt	$f_{xx} f_{yy} - (f_{xy})^2$	f_{xx}	Result
$\left(\frac{1}{3}, \frac{2}{3}\right)$	$+$	$+$	local min.

$$\left.\begin{array}{l} f_{xx}(x,y) = 2 \\ f_{xy}(x,y) = -1 \\ f_{yy}(x,y) = 2 \end{array}\right\} \Rightarrow D = 4 - 1 = 3$$

■

Exp. Let $f(x,y) = x^3 + y^4 - 6x - 2y^2$.

(a) Find all the critical points of f.

(b) Determine if they are local max, local min or saddle points.

Solutions. (a) $\nabla f(x,y) = \langle f_x, f_y \rangle$

$$= \langle 3x^2 - 6, 4y^3 - 4y \rangle$$

To find the critical points we have to solve

$$\nabla f(x,y) = \langle 0, 0 \rangle.$$

$$\begin{cases} 3x^2 - 6 = 0 \\ 4y^3 - 4y = 0 \end{cases} \Rightarrow \begin{cases} x = \pm\sqrt{2} \\ y = 0 \text{ or } y = \pm 1 \end{cases}$$

So there are 6 critical points:

$(-\sqrt{2}, -1)$, $(\sqrt{2}, -1)$, $(-\sqrt{2}, 0)$, $(\sqrt{2}, 0)$, $(-\sqrt{2}, 1)$ and $(\sqrt{2}, 1)$.

(b) We use the 2^{nd} derivative test:

Critical pts	$D = f_{xx} f_{yy} - (f_{xy})^2$	f_{xx}	Result
$(-\sqrt{2}, -1)$	$-$		Saddle pt
$(\sqrt{2}, -1)$	$+$	$+$	local min

$(-\sqrt{2}, 0)$	$+$	$-$	local max
$(\sqrt{2}, 0)$	$-$		$\Longrightarrow$ Saddle pt
$(-\sqrt{2}, 1)$	$-$		$\Longrightarrow$ Saddle pt
$(\sqrt{2}, 1)$	$+$	$+$	local min

$$\left.\begin{aligned} f_{xx}(x,y) &= 6x \\ f_{xy}(x,y) &= 0 \\ f_{yy}(x,y) &= 12y^2 - 4 \end{aligned}\right\} \Longrightarrow D = 24x(3y^2 - 1)$$

Summary: To Find Local Extreme Points

Step 1. Find ∇f.

Step 2. Find the critical pts by solving $\nabla f = \vec{0}$.

Step 3. Compute f_{xx}, f_{xy}, f_{yy}, and

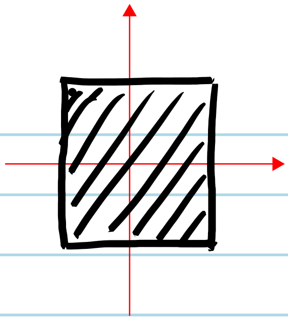
$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} \cdot f_{yy} - (f_{xy})^2,$$

and use the 2^{nd} derivative test.

Global extreme values

- A continuous function always have a global max and a global min in a closed and bounded region.

Exp.



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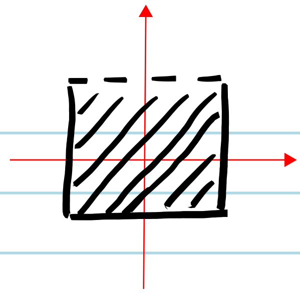

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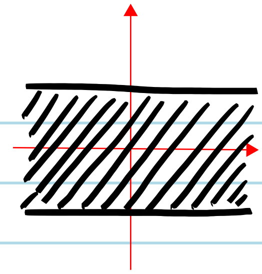

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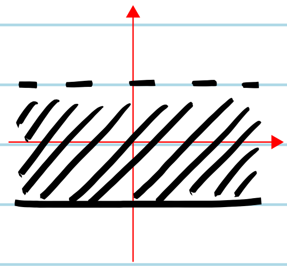

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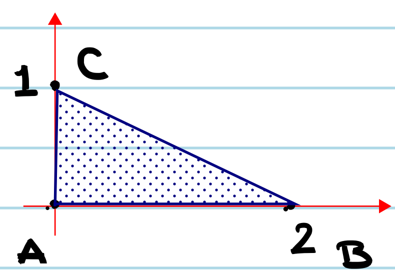
> Extreme values occure either
>
> (1) At a critical point in the interior of D.
>
> or
>
> (2) At a point on the boundary of D.

Exp. Find the global max and min of

[In class, we considered (0,0), (1,0) and (0,1) which has less computation.]

$$f(x,y) = x^3 + y^3 - 3xy$$

in the triangle D.



Solution. Step 1. Find ∇f.

$$\nabla f(x,y) = \langle 3x^2 - 3y \,,\, 3y^2 - 3x \rangle$$

Step 2. Solve $\nabla f = \vec{0}$ to find the critical pts.

$$\begin{cases} 3x^2-3y=0 \\ 3y^2-3x=0 \end{cases} \Rightarrow \begin{cases} y=x^2 \\ y^2=x \end{cases} \Rightarrow (x^2)^2=x$$

$\Rightarrow x^4=x \Rightarrow x(x^3-1)=0 \Rightarrow x=0$ or $x^3=1$

$\Rightarrow x=0$ or $x=1$.

If $x=0$, then $y=x^2=0$

If $x=1$, then $y=x^2=1$.

So there are two critical points $(0,0)$ and $(1,1)$.

Step 3. Find the extreme values on the boundary.

[To this end, either we parametrize the boundary ,or we use Lagrange multipliers. In this example we parametrize each segment.]

Segment AB : $y=0$, $0 \leq x \leq 2$

$g(x)=x^3 \Rightarrow g$ is increasing

$\Rightarrow g$ has a min at $x=0$

and a max at $x=2$.

$\Rightarrow f(0,0)=0$ min of f over AB,
$f(2,0)=8$ max of f over AB.

Segment BC: $\vec{r}(t) = t\,\overrightarrow{OC} + (1-t)\,\overrightarrow{OB}$

$$= t\langle 0,1\rangle + (1-t)\langle 2,0\rangle$$

$$= \langle 2-2t, t\rangle \quad \text{for} \quad 0 \le t \le 1.$$

$$h(t) = f(\vec{r}(t))$$

$$= (2-2t)^3 + t^3 - 3(2-2t)(t)$$

$$= (8 - 24t + 24t^2 - 8t^3) + t^3 - (6t - 6t^2)$$

$$= 8 - 30t + 30t^2 - 7t^3$$

Now we use the techniques of finding extreme values of a single variable function:

$$h'(t) = -30 + 60t - 21t^2$$

$$= -3(10 - 20t + 7t^2)$$

$$h'(t) = 0 \Rightarrow t^2 - \frac{20}{7}t + \frac{10}{7} = 0$$

$$\Rightarrow t^2 - \frac{20}{7}t + \frac{100}{49} = \frac{100}{49} - \frac{10}{7}$$

$$\Rightarrow \left(t - \frac{10}{7}\right)^2 = \frac{30}{49}$$

$$\Rightarrow t = \frac{10 \pm \sqrt{30}}{7}$$

~ 0.75

t	0		$\frac{10-\sqrt{30}}{7}$		1
$h'(t)$		$-$	0	$+$	
$h(t)$	8	$\searrow$	$\frac{292-60\sqrt{30}}{49}$	$\nearrow$	1

$\Rightarrow$ $f\left(\frac{-6+2\sqrt{30}}{7}, \frac{10-\sqrt{30}}{7}\right) = \frac{292-60\sqrt{30}}{49}$

is the min of f on BC, and

$f(2,0) = 8$ is the max of f on BC.

<u>Segment AC</u>. $x=0$, $0 \le y \le 1$

$\ell(y) = f(0,y) = y^3$ which is increasing.

$\Rightarrow$ $f(0,0)=0$ is the min of f on AC

$f(0,1)=1$ is the max of f on AC.

<u>Step 4</u>. Compare the values of f at all the relevant points.

Points	$f(x,y)$
(Critical points in the interior)	
$(1,1)$	-1 min
(Extreme points on each seg. of the boundary)	

$(0,0)$	0
$(2,0)$	8 max
$\left(\frac{-6+2\sqrt{30}}{7}, \frac{10-\sqrt{30}}{7}\right)$	$\frac{292-60\sqrt{30}}{49}$
$(0,1)$	1

So the global max of f is 8 which occurs at $(2,0)$, and

the global min is -1 which occurs at $(1,1)$. ∎

A better way of dealing with extreme values of f at the boundary is using the method of Lagrange multipliers.

Lagrange Multipliers

How can we find the extreme values of $f(x,y)$ under the constraint $g(x,y)=c$? [Three-variable functions are similar.]

Let's recall that the rate of change of f

along a curve $\vec{r}(t)$ is $\vec{r}'(t_0) \cdot \nabla f(P_0)$. So if $f(P_0)$ is an extreme value of f on the curve $\vec{r}(t)$, then $\nabla f(P_0)$ is orthogonal to the curve at P_0.

By the above discussion, if $f(P_0)$ is an extreme value of f on the level curve $g(P)=0$, then $\nabla f(P_0)$ is orthogonal to the level curve of g. Hence it is parallel to $\nabla g(P_0)$.

Summary If $f(P_0)$ is an extreme value of f under the constraint $g(P)=0$, then

$$\begin{cases} \nabla f(P_0) = c\,\nabla g(P_0) & \text{for some } c. \\ g(P_0) = 0 \end{cases}$$

$$\Rightarrow \langle 3,4\rangle = c\langle 2(x-1), 2y\rangle$$

$$\Rightarrow \begin{cases} c = \dfrac{3}{2(x-1)} & (x \text{ cannot be } 1) \\ c = \dfrac{2}{y} & (y \text{ cannot be } 1) \end{cases}$$

$$\Rightarrow \frac{3}{2(x-1)} = \frac{2}{y}$$

$$\Rightarrow y = \frac{4}{3}(x-1)$$

$$\Rightarrow (x-1)^2 + \frac{16}{9}(x-1)^2 = 1$$

$$\Rightarrow (x-1)^2 = \frac{9}{25} \Rightarrow x-1 = \pm\frac{3}{5}.$$

$$\Rightarrow x = \frac{8}{5} \text{ or } \frac{2}{5}.$$

$$x = \frac{8}{5} \Rightarrow y = \frac{4}{3}(x-1) = \frac{4}{5}$$

$$x = \frac{2}{5} \Rightarrow y = \frac{4}{3}(x-1) = -\frac{4}{5}$$

$$\Rightarrow f\left(\frac{8}{5}, \frac{4}{5}\right) = \frac{24}{5} + \frac{16}{5} = \frac{40}{5} = 8 \quad \text{max}$$

$$f\left(\frac{2}{5}, \frac{4}{5}\right) = \frac{6}{5} - \frac{16}{5} = -2 \quad \text{min.}$$

(Notice that, since circle is a closed and bounded region and f is continuous, the max and the min exist. So the Lagrange multipliers method gives us the answer.) $\blacksquare$

Exp. Find the closest point to the origine on the plane $2x+3y+4z=9$.

Sol. $f(x,y,z) = x^2+y^2+z^2$

$g(x,y,z) = 2x+3y+4z$

$\nabla f = c\,\nabla g$

$\Rightarrow \langle 2x, 2y, 2z\rangle = c\langle 2,3,4\rangle$

$\Rightarrow \quad c = x = \frac{2}{3}y = \frac{z}{2}$

$\Rightarrow \quad 2x + 3\left(\frac{3}{2}x\right) + 4(2x) = 1$

$\Rightarrow (4+9+16)x = 2$

$\Rightarrow x = \dfrac{2}{29}.$

$\Rightarrow y = \left(\frac{3}{2}\right)\left(\frac{2}{29}\right) = \frac{3}{29}$ and $z = (2)\left(\frac{2}{29}\right) = \frac{4}{29}$

$\Rightarrow$ The closest point is $\left(\frac{2}{29}, \frac{3}{29}, \frac{4}{29}\right)$.

(Notice that by geometric considerations it is clear that the min. distance exists and the max does not exist. So the Lagrange multiplier method gives us the min.) ■

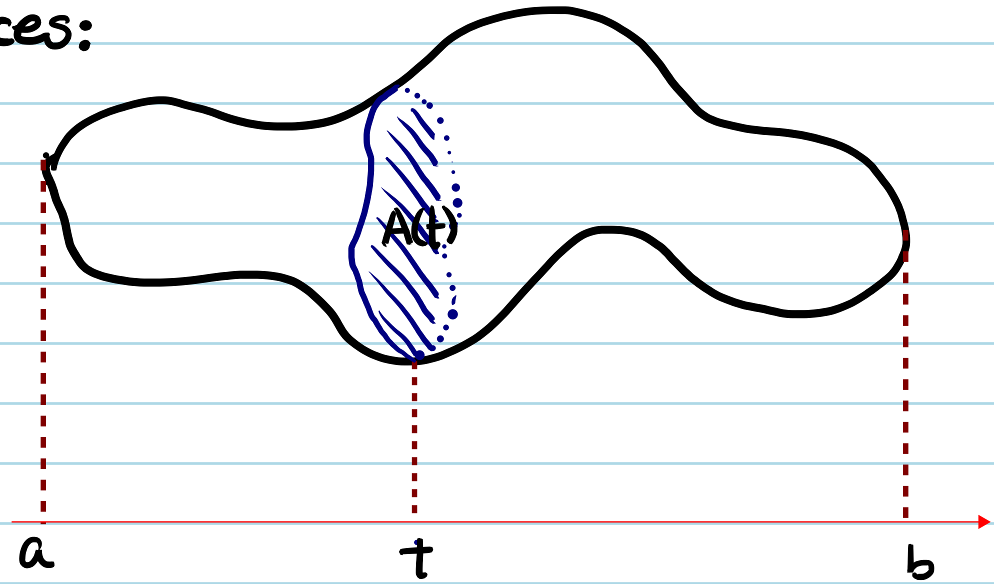
Double Integrals

Let D be a (nice) region in the xy-plane, and f be a (nice) function, e.g. continuous, defined on D. Suppose f is non-negative. Then the double integral is defined in a way that gives us

$$\boxed{\iint_D f(x,y)\,dA = \text{volume of the solid above } D \text{ and under the graph } z=f(x,y) \text{ of } f.}$$

How can we compute a double integral?

More or less the idea goes back to Greeks. By virtue of Cavalieri's Principle, in order to compute the volume of a solid, one needs to find the areas of its slices:

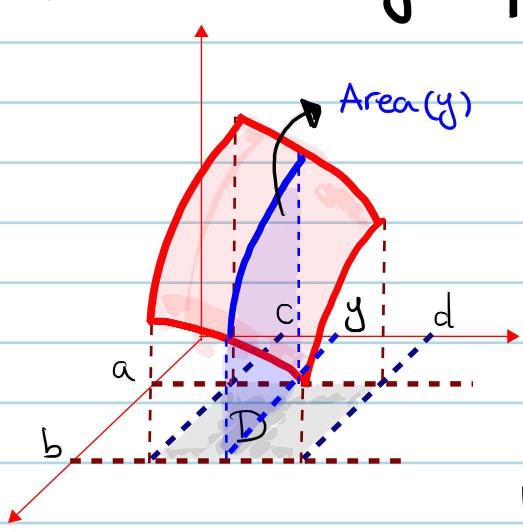


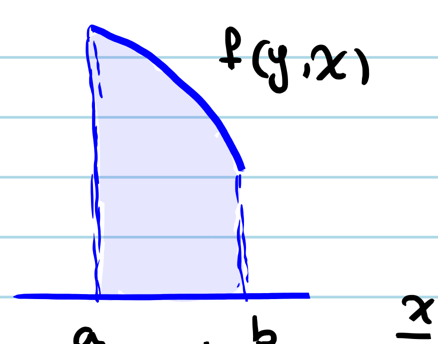
$$\text{volume} = \int_a^b \text{area}(t)\, dt.$$

Now suppose E is the solid above the rectangle

$a \leq x \leq b$ and $c \leq y \leq d$, and

under the graph $z = f(x,y)$ of f. Then we can slice E using the planes parallel to the xz-plane, or the yz-plane:





$$\text{Area}(y) = \int_a^b f(x,y)\, dx$$

[y should be treated as

a constant.]

$$\Rightarrow \iint_D f(x,y)\,dA = \text{volume of } E$$
$$= \int_c^d \text{Area}(y)\,dy$$
$$= \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

By a similar argument (this time using planes parallel to the yz-plane) we have

$$\iint_D f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

Summary [Fubini] Let D be the rectangle $a \le x \le b$, $c \le y \le d$, and f be a continuous fun.

Then
$$\iint_D f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$
$$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Warning and Suggestion Make sure that the order of the intervals is correct. First decide in which ordering you would like to integrate, and then write down the intervals:

First $\int\int f(x,y)\, dx\, dy$

Then $\int \int f(x,y)\, dx\, dy$

Exp. [Geometric Understanding of Double Integral.]

Find $\iint_D dA$, where D is a nice region in the xy-plane.

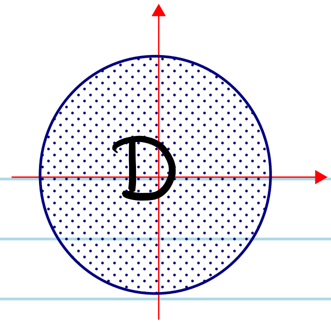
Solution. It is the volume of a cylindricall solid with base D and height 1. So it is

height $\times$ area of the base. Hence

$$\iint_D dA = \text{area of } D.$$

Exp. [Geometric Understanding of Double Integral.]

Find $\iint_D \sqrt{1-x^2-y^2}\, dA$ where D is the unit disk centered at the origine.



<u>Solution</u>. $\iint_D \sqrt{1-x^2-y^2}\, dA =$ volume of the solid above D and under $z = \sqrt{1-x^2-y^2}$.

$$z = \sqrt{1-x^2-y^2} \Rightarrow z^2 = 1 - x^2 - y^2$$

$$\Rightarrow x^2 + y^2 + z^2 = 1$$

which is a sphere of radius 1 centered at the origine.

So the solid above D and under $z = \sqrt{1-x^2-y^2}$ is half a ball of radius 1. Hence

$$\iint_D \sqrt{1-x^2-y^2}\, dA = \frac{1}{2} \cdot \frac{4}{3}\pi = \frac{2}{3}\pi.$$

Solution.

$$\iint_D xy\, dA = \int_1^2 \int_0^1 xy \; dy\, dx$$

$$\boxed{A(x) = \int_0^1 xy\, dy = \frac{xy^2}{2}\Big|_0^1 = \frac{1}{2}x}$$

$$= \int_1^2 \frac{1}{2}x\, dx$$

$$= \frac{x^2}{4}\Big|_1^2 = 1 - \frac{1}{4} = \frac{3}{4}.$$

Exp. [Sometimes it is better to change the order of integration.]

Exp. $\int_0^1 \int_0^{\pi/2} x \cos(xy)\, dx\, dy = ?$

1

Solution. With the given order of integration, we need to compute $\int_0^{\pi/2} x \cos(xy)\, dx$. This can be done using integaration-by-part. This, however, is not the easiest way to compute the given integral.

Using Fubini, we have

$$\int_0^1 \int_0^{\pi/2} x \cos(xy)\, dx\, dy = \int_0^{\pi/2}\int_0^1 x \cos(xy)\, dy\, dx$$

In this order of integration, we need to find $\int_0^1 x\cos(xy)\,dy$, which can be easily done

using the substitution rule.

$$A(x) = \int_0^1 x \cos(xy)\, dy = \int_0^x \cos(u)\, du$$

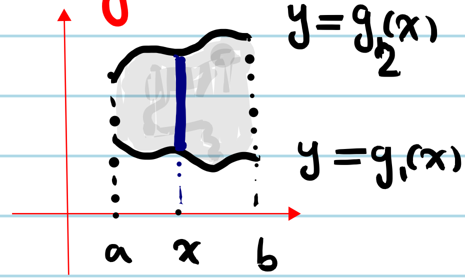
$$\boxed{u = xy \Rightarrow du = x\, dy}$$

$$= \sin(u)\Big|_0^x = \sin(x).$$

$$\Rightarrow \int_0^{\pi/2}\int_0^1 x\cos(xy)\, dy\, dx = \int_0^{\pi/2} \sin(x)\, dx$$

$$= -\cos(x)\Big|_0^{\pi/2} = 1.$$

What if D is NOT a rectangle?

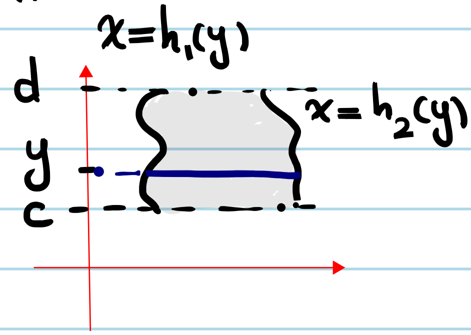


Vertically simple

$$a \le x \le b$$

and $g_1(x) \le y \le g_2(x)$

$$\Rightarrow \iint_D f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$



Horizontally simple

$$c \le y \le d$$

and $h_1(y) \le x \le h_2(y)$

$$\Rightarrow \iint_D f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

Warning. The boundary of integration cannot depend on the variable with respect to which you are integrating.

$$\int_{g_1(\underline{\underline{x}})}^{g_2(\underline{\underline{x}})} f(x,y)\, \cancel{dx}$$

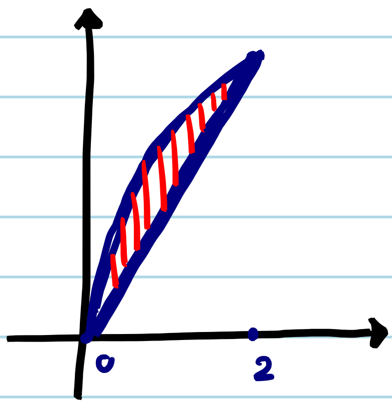
Exp. $0 \le x \le 2$

$\frac{3}{2}x \le y \le 3\sqrt{\frac{x}{2}}$

(1) Sketch the region D.

(2) $\iint_D dA = ?$

Solution.



$$\iint_D dA = \int_0^2 \int_{\frac{3}{2}x}^{3\sqrt{\frac{x}{2}}} dy\, dx$$

$$A(x) = \int_{\frac{3}{2}x}^{3\sqrt{\frac{x}{2}}} dy = y \Big|_{\frac{3}{2}x}^{3\sqrt{\frac{x}{2}}} = 3\sqrt{\frac{x}{2}} - \frac{3}{2}x$$

$$\iint_D dA = \int_0^2 3\sqrt{\frac{x}{2}} - \frac{3}{2}x \, dx$$

$$= \left(\frac{3}{\sqrt{2}} \cdot \frac{2}{3} \cdot x^{3/2} - \frac{3}{2} \cdot \frac{1}{2} \cdot x^2\right)\Big|_0^2$$

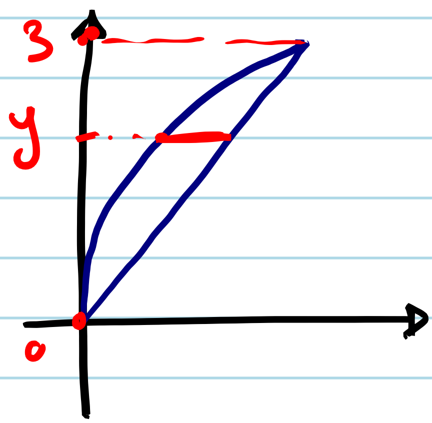
$$= \sqrt{2} \cdot \sqrt{2}^{\,3} - \frac{3}{4} \cdot 4$$

$$= 4 - 3 = 1.$$

$\blacksquare$

Exp. Notice that the above region is also a horizontally simple region. Setup the integral:

$$\int_0^3 \int_{\frac{2}{9}y^2}^{\frac{2}{3}y} 1 \quad dx\, dy$$



$$y = 3\sqrt{\frac{x}{2}} \Rightarrow \left(\frac{y}{3}\right)^2 = \frac{x}{2}$$

$$\Rightarrow \quad x = \frac{2}{9}y^2$$

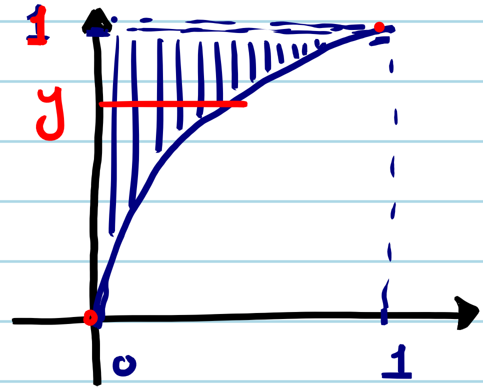
$$y = \frac{3}{2}x \Rightarrow x = \frac{2}{3}y$$

Sometimes we have to choose the order carefully:

Exp. $\int_0^1 \int_{x^{2/3}}^1 x e^{y^4}\, dy\, dx$

- Sketch the region.
- Change the order.
- Compute.

Solution.



$$\int_0^1 \int_0^{y^{3/2}} x\, e^{y^4} \, dx\, dy$$

$$y = x^{2/3} \Rightarrow x = y^{3/2}$$

$$A(y) = \int_0^{y^{3/2}} x\, e^{y^4}\, dx = e^{y^4} \frac{x^2}{2} \Big|_0^{y^{3/2}}$$

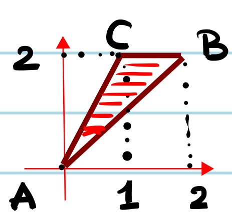
$$= \frac{1}{2} y^3 e^{y^4}.$$

$$\iint_D x e^{y^4}\, dA = \int_0^1 \frac{1}{2} y^3 e^{y^4}\, dy = \int_0^1 \frac{1}{2} \cdot \frac{1}{4} e^u\, du$$

$$\boxed{u = y^4 \Rightarrow du = 4y^3\, dy}$$

$$= \frac{1}{8} e^u \Big|_0^1 = \frac{1}{8}(e-1).$$

Exp. Find the volume of the solid above the triangle D, and under $z = x + y$.



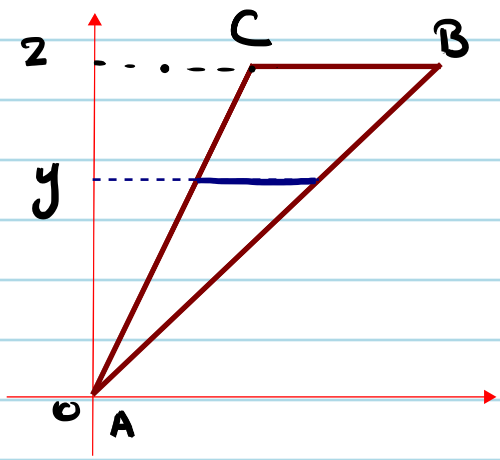
Solution

Since $\iint_D f(x,y)\, dA =$ volume of the solid above D and under $z = f(x,y)$,

we have

$$\text{volume} = \iint_D x+y \; dA \; .$$

Region D can be considered as a horizontally simple region.



Equation of AC : $y = 2x \Rightarrow x = \frac{1}{2}y$

Equation of AB : $y = x \Rightarrow x = y$

So D is $\quad 0 \le y \le 2$ and $\frac{1}{2}y \le x \le y$.

Hence

$$\text{volume} = \int_0^2 \int_{\frac{1}{2}y}^{y} x+y \quad dx\, dy$$

$$A(y) = \int_{\frac{1}{2}y}^{y} x+y \, dx = \left(\frac{1}{2}x^2 + yx\right)\Big|_{\frac{1}{2}y}^{y}$$

$$= \left[\left(\frac{1}{2}y^2 + y^2\right) - \left(\frac{1}{2}\left(\frac{1}{2}y\right)^2 + y\left(\frac{1}{2}y\right)\right)\right]$$

$$= \left[\frac{3}{2}y^2 - \left(\frac{1}{8}y^2 + \frac{1}{2}y^2\right)\right]$$

$$= \frac{7}{8}y^2.$$

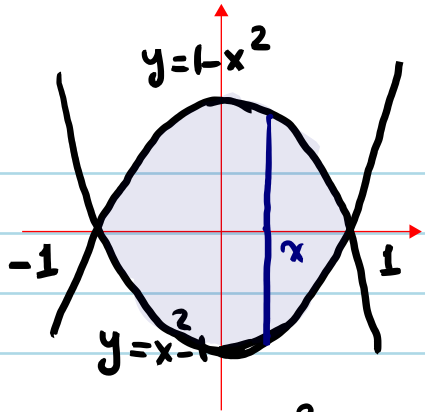
$$\text{volume} = \int_0^2 \frac{7}{8}y^2 \, dy = \frac{7}{8} \cdot \frac{1}{3}y^3 \Big|_0^2$$

$$= \frac{7}{8} \cdot \frac{1}{3} \cdot 8 = \frac{7}{3}.$$

Exp. Find the volume of the solid above the region D enclosed by $y = x^2 - 1$ and $y = 1 - x^2$, and under $z = x^2$.

Solution.

- volume $= \iint_D x^2 \, dA$
- (Sketch the region D.)



$-1 \leq x \leq 1$

$x^2-1 \leq y \leq 1-x^2$

- volume $= \displaystyle\int_{-1}^{1}\int_{x^2-1}^{1-x^2} x^2 \, dy \, dx$

$$A(x) = \int_{x^2-1}^{1-x^2} x^2 \, dy = x^2 y \Big|_{x^2-1}^{1-x^2}$$

$$= x^2\left[(1-x^2)-(x^2-1)\right]$$

$$= 2x^2 - 2x^4$$

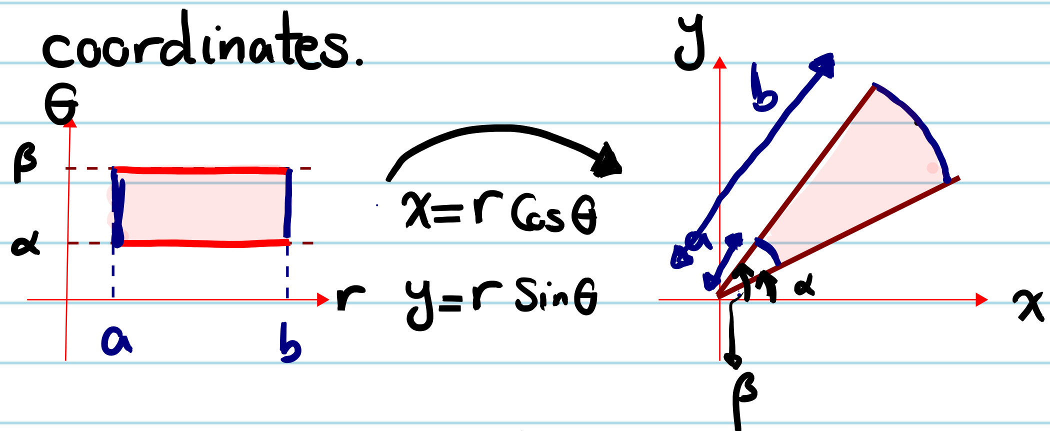
$$\text{volume} = \int_{-1}^{1} 2x^2 - 2x^4 \, dx$$

$$= \left(\frac{2}{3}x^3 - \frac{2}{5}x^5\right)\Big|_{-1}^{1}$$

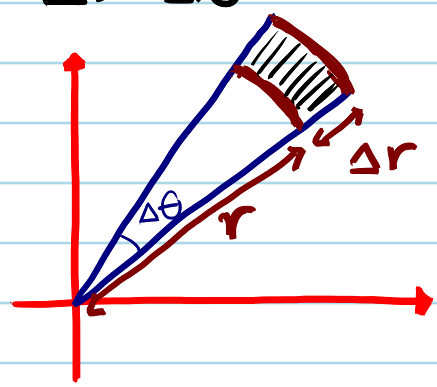
$$= \frac{8}{15}$$

Polar Coordinates

If the region of integration is angular, it is NOT easy to describe it in terms of vertically or horizontally simple regions. Instead we use polar coordinates.



If r changes a bit to $r + \Delta r$ and θ changes to $\theta + \Delta\theta$, A changes to $A + \Delta A$ where $\Delta A \approx r \, \Delta r \, \Delta\theta$



$$\Delta A = \frac{1}{2}(r+\Delta r)^2 \Delta\theta - \frac{1}{2} r^2 \Delta\theta$$

$$= \frac{1}{2}\left[r^2 + 2r \cdot \Delta r + \Delta r^2 - r^2\right] \Delta\theta$$

$$= r \cdot \Delta r \, \Delta\theta + \frac{1}{2} \Delta r^2 \cdot \Delta\theta$$

$$\approx r \cdot \Delta r \cdot \Delta\theta$$

$$\Rightarrow \quad dA = dx \, dy = r \, dr \, d\theta$$

Something to remember the $\underline{\underline{r}}$ factor in $\underline{\underline{r\,dr\,d\theta}}$:

dA is of "dimension 2".

dx, dy and dr are of "dimension 1".

Angle is of "dimension 0"

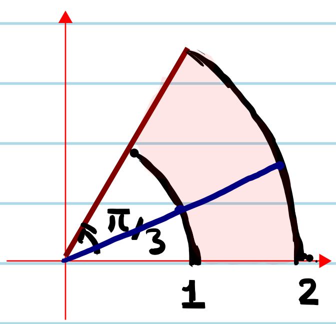
So $\underbrace{dA}_{2}$ cannot be $\underbrace{dr}_{1}\,\underbrace{d\theta}_{0}$.

Summary Let D be the region

$$\alpha \le \theta \le \beta \quad \text{and} \quad a \le r \le b.$$

Then $$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Exp. Find $\iint_D x\,dA$ where



Solution Describe D in polar coordinates. Always start with θ: $0 \le \theta \le \pi/3$ and $1 \le r \le 2$.

$$\iint_D x\,dA = \int_0^{\pi/3}\int_1^2 (r\cos\theta)\; r\,dr\,d\theta$$

$$A(\theta) = \int_1^2 r^2 \cos\theta \, dr = \frac{1}{3} r^3 \cos\theta \Big|_1^2$$

$$= \frac{7}{3} \cos\theta .$$

$$\iint_D x \, dA = \int_0^{\pi/3} \frac{7}{3} \cos\theta \, d\theta$$

$$= \frac{7}{3} \sin\theta \Big|_0^{\pi/3} = \frac{7\sqrt{3}}{6} .$$

<u>Exp.</u> Find the volume of a solid between the xy-plane and $z = 4 - x^2 - y^2$.

<u>Solution</u>. <u>Step 1.</u> Sketch the solid.

<u>Step 2.</u> Find its projection onto the xy-plane. [Region D]

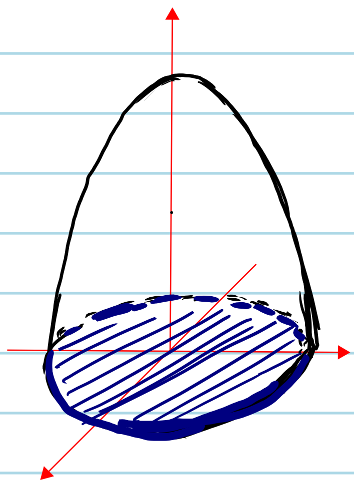
<u>Step 3.</u> Setup the integral.

<u>Step 4.</u> Decide what the best way is to describe D:

vertically simple, horizontally simple or polar coordinates.

Step 5. Setup the iterated integrals.

Step 6. Compute.

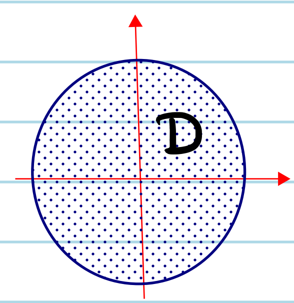


The intersection of $z=0$ and $z=4-x^2-y^2$ is

$0=4-x^2-y^2$ in the xy-plane

$\Rightarrow$ $x^2+y^2=4$ in the xy-plane

So its shadow in the xy-plane is the disk D of radius 2 centered at the origine.



$$\text{volume} = \iint_D 4-x^2-y^2 \, dA$$

In polar coordinates: $0 \le \theta \le 2\pi$ and $0 \le r \le 2$

$$\text{volume} = \int_0^{2\pi}\int_0^2 (4-r^2)\, r \, dr \, d\theta$$

$$A(\theta) = \int_0^2 (4-r^2)\, r\, dr = \int_0^2 4r-r^3 \, dr$$

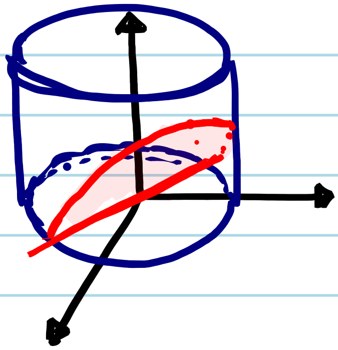
$$= \left(2r^2 - \frac{r^4}{4}\right)\Big|_0^2 = 8-4=4$$

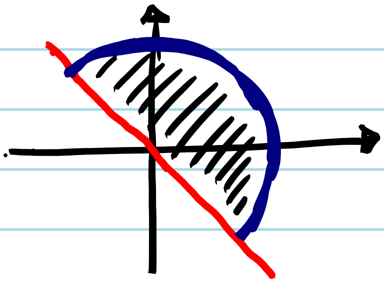
$$\text{volume} = \int_0^{2\pi} 4 \, d\theta = 8\pi.$$

Exp. Find the volume of the solid

- cylinder $x^2+y^2=4$ (in)
- plane $z = x+y$ (under)
- plane xy (above)

Sol. • Sketch the graphs to understand the projection of the solid onto xy-plane.





. Write volume in terms of a double integral

$$\text{vol.} = \iint_D x+y \, dA$$

. Describe D in polar coordinates:

$$-\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4} \quad , \quad 0 \leq r \leq 2$$

$$\text{. vol.} = \int_{-\pi/4}^{3\pi/4} \int_0^2 (r\cos\theta + r\sin\theta)\, r\, dr\, d\theta$$

$$A(\theta) = \int_0^2 r^2 (\cos\theta + \sin\theta)\, dr$$

$$= \frac{8}{3}(\cos\theta + \sin\theta).$$

$$\Longrightarrow \text{vol.} = \int_{-\pi/4}^{3\pi/4} \frac{8}{3}(\cos\theta + \sin\theta)\, d\theta$$

$$= \frac{8}{3}(\sin\theta - \cos\theta)\Big|_{-\pi/4}^{3\pi/4}$$

$$= \frac{8}{3}\left[\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - \left(\frac{-\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)\right]$$

$$= \frac{16}{3}\sqrt{2}. \qquad \blacksquare$$