Summary of the 6th and the 7th weeks's lectures

In the previous week we saw that if $z=f(x,y)$ has a tangent plane at $(a,b,f(a,b))$, then it is the graph of

$$L(x,y)=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b).$$

<u>Definition</u> f is called differentiable at (a,b) if $z=f(x,y)$ has a tangent plane at $(a,b,f(a,b))$.

<u>Alternatively</u> if

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)-L(x,y)}{\sqrt{(x-a)^2+(y-b)^2}}=0,$$

where $L(x,y)=c+e(x-a)+f(y-b)$ for some constants c, e and f.

In class we did not discuss this, but I include it here for you to read:

If f is differentiable at (a,b), then the partial derivatives of f with respect to x and y

exist, and the function L in the definition of differentiability is uniqued, and it is

$$f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Criteria of differentiability

If f_x and f_y exist and are continuous at an open disk D, then f is differentiable at D.

Exp. (a) At which points is $f(x,y) = \sqrt{x^2+y^2}$ non-differentiable ?

(b) Find an equation of the tangent plane of $z = f(x,y)$ at $(1, -1, \sqrt{2})$.

Solution (a) $f_x = \dfrac{x}{\sqrt{x^2+y^2}}$ and $f_y = \dfrac{y}{\sqrt{x^2+y^2}}$

So f_x and f_y exist and continuous at any point other than (0,0). If $p \neq (0,0)$, then f_x and f_y exist and are continuous at the open disk centered at p with radius $|Op|/2$

(in fact any radius $< |OP|$ works); So by the above criteria f is differentiable at p.

On the other hand the partial derivative of f with respect to x does NOT exist at $(0,0)$ and so f is NOT differentiable at $(0,0)$.

$f_x(0,0)$ does NOT exist as

$$\lim_{\Delta x \to 0} \frac{f(0+\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{\Delta x^2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x} \text{ does NOT exist.}$$

(Notice that $\displaystyle\lim_{\Delta x \to 0^+} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{\Delta x}{\Delta x} = 1$

and $\displaystyle\lim_{\Delta x \to 0^-} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-\Delta x}{\Delta x} = -1$.)

(b) Since f is differentiable at $(1,-1)$, its graph has a tangent plane. It can be given by the following equation:

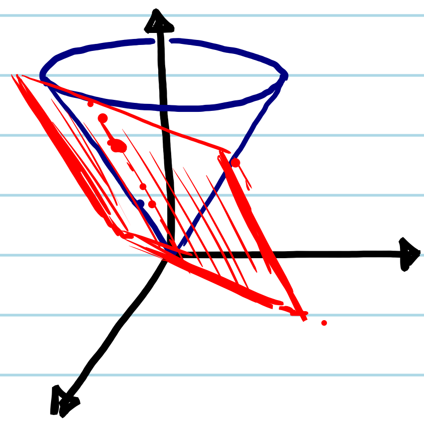
$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

- $f_x(1,-1) = \frac{1}{\sqrt{2}}$ and $f_y(1,-1) = \frac{-1}{\sqrt{2}}$
- $f(1,-1) = \sqrt{2}$.

So $z = \sqrt{2} + \frac{1}{\sqrt{2}}(x-1) - \frac{1}{\sqrt{2}}(y+1)$

$\Rightarrow z = \sqrt{2} - \frac{2}{\sqrt{2}} + \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}$

$\Rightarrow z = \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}$.



Exp. Find the points on the graph $z = x^2 - y^2$ at which $\vec{n} = \langle 3, 1, 2 \rangle$ is normal to the tangent plane

Solution. We know that an equation of the tangent plane at $(a, b, f(a,b))$ is

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

In particular $\langle -f_x(a,b), -f_y(a,b), 1 \rangle$

is a normal vector. If $\vec{n} = \langle 3, 1, 2 \rangle$ is also a normal vector, then for a constant c we have $\vec{n} = c \langle -f_x(a,b), -f_y(a,b), 1 \rangle$.

In our case we have $f_x(x,y) = 2x$ and $f_y(x,y) = -2y$ and so

$$\langle 3, 1, 2 \rangle = c \langle -2a, 2b, 1 \rangle$$

$$\Rightarrow \left.\begin{aligned} 3 &= -2ac \\ 1 &= 2bc \\ 2 &= c \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} c &= 2 \\ a &= \frac{-3}{4} \\ b &= \frac{1}{4} \end{aligned}\right.$$

So $\vec{n}$ is a normal vector of the tangent plane only at $\left(\frac{-3}{4}, \frac{1}{4}, \frac{1}{2}\right)$

(<u>Notice</u> : $z = x^2 - y^2 = \frac{9}{16} - \frac{1}{16} = \frac{1}{2}$.)

<u>Approximate values of f</u>

$$f(a+\Delta x, b+\Delta y) \approx L(a+\Delta x, b+\Delta y)$$

$$= f(a,b) + f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y$$

Alternatively

$$\Delta z \approx f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y.$$

$$= \langle \Delta x, \Delta y\rangle \cdot \underbrace{\langle f_x(a,b), f_y(a,b)\rangle}_{\text{The gradient of } f \text{ at } (a,b).}.$$

Definition. For a two variable function f, the gradient $\nabla f(x,y)$ of f at (x,y) is

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle.$$

If f is a function of x, y, and z, then

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle.$$

So $\Delta z \approx \langle \Delta x, \Delta y\rangle \cdot \nabla f(a,b).$

What is the rate of change of as we move a curve given by the vector parametrization $\vec{r}(t)$?

This means what is $\frac{d}{dt} f(\vec{r}(t))$?

$$\frac{d}{dt} f(\vec{r}(t)) = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\langle \Delta x, \Delta y \rangle \cdot \nabla f(\vec{r}(t))}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left\langle \frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t} \right\rangle \cdot \nabla f(\vec{r}(t))$$

$$= \langle x'(t), y'(t) \rangle \cdot \nabla f(\vec{r}(t))$$

$$= \vec{r}\,'(t) \cdot \nabla f(\vec{r}(t)).$$

$\boxed{\text{Chain Rule For Curves}}$ $\quad \frac{d}{dt} f(\vec{r}(t)) = \vec{r}\,'(t) \cdot \nabla f(\vec{r}(t))$

If $\vec{r}(t)$ is part of a level curve (or is in a level surface), then $f(\vec{r}(t))$ is constant. So $\frac{d}{dt} f(\vec{r}(t)) = 0$, which implies

$\nabla f(p)$ is orthogonal to the level curve (or level surface) which passes through p.

What is the rate of change of f at P_0 at the direction of $\vec{v}$?

Whenever we talk about "direction", it is better to take the unit vector $\vec{u}$ in the direction of $\vec{v}$.

So $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$.

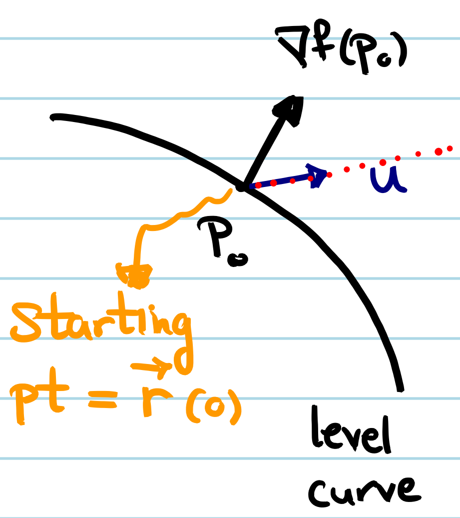
The above question asks us to find the rate of change of f as we move along a line that passes through P_0 with constant velocity $\vec{u}$.

So $\vec{r}(t) = \overrightarrow{OP_0} + t\,\vec{u}$ and $\vec{r}\,'(t) = \vec{u}$.

Hence by the Chain Rule For Curves we have

$$\frac{d}{dt} f(\vec{r}(t)) = \vec{r}\,'(t) \cdot \nabla f(\vec{r}(t))$$
$$= \vec{u} \cdot \nabla f(\vec{r}(t))$$

Since we are interested at the rate of change at P_0, we have to plug in $t=0$.



Hence

$$\boxed{\begin{array}{l}\text{The rate of change}\\ \text{of } f \text{ at } p_0 \text{ in the}\\ \text{direction of the}\\ \text{unit vector } \vec{u}\end{array} = \vec{u}\cdot\nabla f(p_0)}$$

$$\boxed{\begin{array}{l}\text{This is also called the directional derivative}\\ \text{of } f \text{ at } p_0 \text{ in the direction of } \vec{u}. \text{ And}\\ \text{it is denoted by } D_{\vec{u}} f(p_0)\end{array}}$$

So

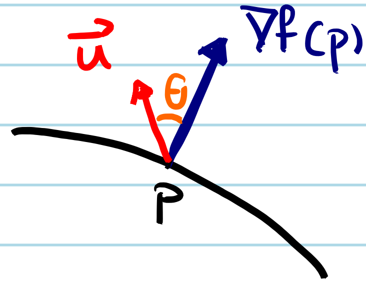
The directional derivative of f at p_0 in the direction of a vector $\vec{v}$ (which is not necessarily unit) $= \dfrac{\vec{v}\cdot\nabla f(p_0)}{\|\vec{v}\|}$

And

$$D_{\vec{u}} f(p) = \vec{u}\cdot\nabla f(p)$$

$$= \|\vec{u}\|\,\|\nabla f(p)\|\cos\theta$$

So $\quad D_{\vec{u}} f(p) = \|\nabla f(p)\| \cos\theta$



$\Rightarrow -\|\nabla f(p)\| \le D_{\vec{u}} f(p) \le \|\nabla f(p)\|$

equality holds iff $\cos\theta = -1$ iff in the direction of $-\nabla f(p)$.

equality holds iff $\cos\theta = 1$ iff in the direction of $\nabla f(p)$

Summary

(1) f increases in the direction of $\vec{v}$ at p if the angle between $\vec{v}$ and $\nabla f(p)$ is <u>acute</u> if $\vec{v} \cdot \nabla f(p) > 0$.

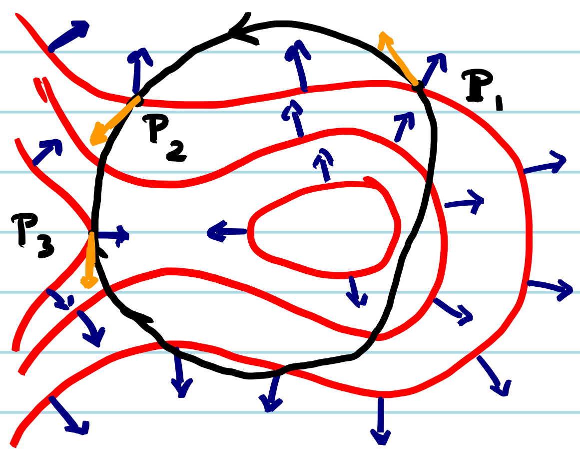
(2) f decreases in the direction of $\vec{v}$ at p if the angle between $\vec{v}$ and $\nabla f(p)$ is <u>obtuse</u> if $\vec{v} \cdot \nabla f(p) < 0$.

(3) The maximum rate of increase of f at p is $\|\nabla f(p)\|$, and the max. rate of increase happens in the direction of $\nabla f(p)$.

(4) The maximum rate of decrease of f at p is $-\|\nabla f(p)\|$, and it happens in the direction of $-\nabla f(p)$.

(5) $D_{\vec{u}} f(p) = \vec{u} \cdot \nabla f(p) = \|\nabla f(p)\| \cos\theta$

where $\vec{u}$ is a unit vector and θ is the angle between $\vec{u}$ and $\nabla f(p)$.



Exp. Red curves are level curves

Blue vectors are gradient

We are moving counter clockwise along the black curve.

Then at P_1, f is increasing as the angle between the tangent vector and the gradient is acute.

At P_2, f is decreasing as the angle between the tangent vector and the gradient is obtuse.

At P_3, f does not increase or decrease as the black curve is tangent to the level curve at P_3.

Tangent plane of a level surface

One of the nice applications of the fact that $\nabla f(P)$ is orthogonal to the level surfaces is in finding an equation of tangent planes:

General Form

Find an equation of the tangent plane of

$$f(x,y,z)=c$$

at $P_0=(x_0,y_0,z_0)$.

Solution. normal vector : $\nabla f(P_0)$

point : P_0

So $\quad \nabla f(P_0) \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$

$\Rightarrow f_x(P_0)\, x + f_y(P_0)\, y + f_z(P_0) = f_x(P_0)\, x_0 + f_y(P_0)\, y_0 + f_z(P_0)\, z_0$

$\blacksquare$

<u>Exp</u>. Find an equation of the tangent plane of

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 - z^2 = 1$$

at $(2,3,1)$.

<u>Solution</u>. This is a level surface of the function

$$f(x,y,z) = \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 - z^2.$$

So $\nabla f(P_0)$ is a normal vector.

$$\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle$$

$$= \left\langle \frac{x}{2}, \frac{2y}{9}, -2z \right\rangle$$

$$\Rightarrow \nabla f(2,3,1) = \left\langle 1, \frac{2}{3}, -2 \right\rangle.$$

$\Rightarrow$ The tangent plane is

$$x + \frac{2}{3}y - 2z = 2 + \left(\frac{2}{3}\right)(3) - 2(1)$$

$$\Rightarrow \quad x + \frac{2}{3}y - 2z = 2.$$

Chain Rule for two or more variables

The chain rule for curves gives us a rule to find the rate of change of $f(x,y)$ when x and y are functions of t. What if x and y are multivariable functions?

$$\left.\begin{array}{l} z = f(x,y) \\ x = x(u,v) \\ y = y(u,v) \end{array}\right\} \Rightarrow$$

z can be viewed as a function of u and v:

$$z = f(x(u,v), y(u,v))$$

How can we compute $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$?

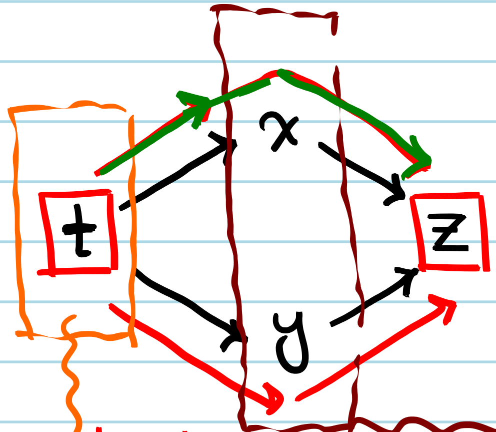
As we saw before, in order to compute a partial derivative, we have to treat other variables as "constants". So in order to compute $\frac{\partial z}{\partial u}$, we treat v as a contant. Hence we can use the chain rule for curves. And we get

$$\frac{\partial z}{\partial u} = \frac{\partial x}{\partial u} \cdot \frac{\partial z}{\partial x} + \frac{\partial y}{\partial u} \cdot \frac{\partial z}{\partial y}$$

and $$\frac{\partial z}{\partial v} = \frac{\partial x}{\partial v} \cdot \frac{\partial z}{\partial x} + \frac{\partial y}{\partial v} \cdot \frac{\partial z}{\partial y}$$

The following diagrams can help us to remember these formulas.

Chain Rule For Curves



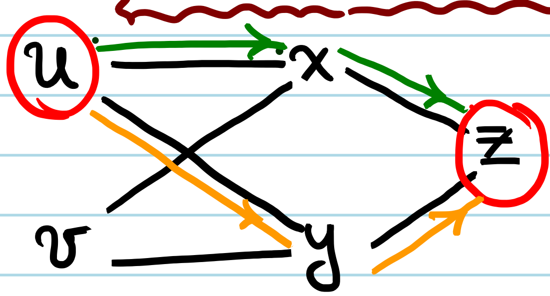
All the paths from t to z:

$$\frac{dz}{dt} = \frac{dx}{dt} \cdot \frac{\partial z}{\partial x} + \frac{dy}{dt} \cdot \frac{\partial z}{\partial y}$$

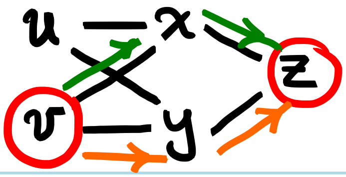
Only one variable in this column. So it is d and NOT ∂

There are two variables in this column: So we have to use ∂ and NOT d

Chain Rule More Variables



$$\frac{\partial z}{\partial u} = \frac{\partial x}{\partial u} \cdot \frac{\partial z}{\partial x} + \frac{\partial y}{\partial u} \cdot \frac{\partial z}{\partial y}$$



$$\frac{\partial z}{\partial v} = \frac{\partial x}{\partial v} \cdot \frac{\partial z}{\partial x} + \frac{\partial y}{\partial v} \cdot \frac{\partial z}{\partial y}.$$

Exp. $z = \cos(x-y)$; $x = 2u + v$ and

$y = u - 2v$.

Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

Solution. $$\frac{\partial z}{\partial u} = \frac{\partial x}{\partial u} \cdot \frac{\partial z}{\partial x} + \frac{\partial y}{\partial u} \cdot \frac{\partial z}{\partial y}$$

$$= (2)(-\sin(x-y)) + (1)(\sin(x-y))$$

$$= -\sin(x-y)$$

$$\Rightarrow \quad \frac{\partial z}{\partial u} = -\sin[(2u+v)-(u-2v)]$$

$$= -\sin(u+3v)$$

Your answer should be written in terms of new variables unless you are told otherwise.

$$\frac{\partial z}{\partial v} = \frac{\partial x}{\partial v} \cdot \frac{\partial z}{\partial x} + \frac{\partial y}{\partial v} \cdot \frac{\partial z}{\partial y}$$

$$= (1)(-\sin(x-y)) + (-2)(\sin(x-y))$$

$$= -3\sin(x-y) = -3\sin(u+3v).$$

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Exp. $z = xy$; $x = r\cos\theta$, $y = r\sin\theta$.

Find $\dfrac{\partial^2 z}{\partial \theta^2}$.

Solution. $$\frac{\partial z}{\partial \theta} = \frac{\partial x}{\partial \theta} \cdot \frac{\partial z}{\partial x} + \frac{\partial y}{\partial \theta} \cdot \frac{\partial z}{\partial y}$$

$$= (-r\sin\theta)(y) + (r\cos\theta)(x)$$

$$= (-r\sin\theta)(r\sin\theta) + (r\cos\theta)(r\cos\theta)$$

$$= r^2(\cos^2\theta - \sin^2\theta) = r^2\cos(2\theta)$$

$$\Rightarrow \frac{\partial z}{\partial \theta} = r^2\cos(2\theta) .$$

$$\Rightarrow \frac{\partial^2 z}{\partial \theta^2} = -2r^2\sin(2\theta) . \qquad \blacksquare$$

Sometimes we do not have an explicit formula for $z = f(x,y)$. We only know an implicit formula

$$F(x,y,z) = 0 .$$

In this case how can we compute $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$?

Let $\omega = F(x,y,z(x,y))$. So ω is a function of x and y. Hence

$$\frac{\partial w}{\partial x} = \frac{\partial x}{\partial x} \cdot \frac{\partial F}{\partial x} + \frac{\partial y}{\partial x} \cdot \frac{\partial F}{\partial y} + \frac{\partial z}{\partial x} \cdot \frac{\partial F}{\partial z}$$

$$\Rightarrow \quad 0 = F_x + \frac{\partial z}{\partial x} F_z$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

Similarly we have $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z}$.

In the lecture, I gave an alternative explanation:

$\nabla F(P_0) = \langle F_x(P_0), F_y(P_0), F_z(P_0) \rangle$ is a normal vector of the tangent plane of $F(x,y,z)=0$ at the point P_0.	$\langle -f_x(a,b), -f_y(a,b), 1 \rangle$ is a normal vector of the tangent plane of $z = f(x,y)$ at the point $(a,b,f(a,b))$.

So, if we know $F(x,y,f(x,y)) = 0$, then the graph $z = f(x,y)$ of f is part of the level surface $F(x,y,z) = 0$. And so the have the same tangent planes at $(a,b,f(a,b))$. Since any two normal vectors of a plane are parallel, we have that $\nabla F(a,b,f(a,b))$ and $\langle -f_x(a,b), -f_y(a,b), 1\rangle$ are parallel. So for some number c we have

$$\langle F_x(a,b,f(a,b)), F_y(a,b,f(a,b)), F_z(a,b,f(a,b))\rangle = c\langle -f_x(a,b), -f_y(a,b), c\rangle.$$

So
$$f_x(a,b) = -\frac{F_x(a,b,f(a,b))}{F_z(a,b,f(a,b))}$$
and
$$f_y(a,b) = -\frac{F_y(a,b,f(a,b))}{F_z(a,b,f(a,b))}.$$

Exp. We know that $z=f(x,y)$ satisfies the following equation:

$$\cos x + \ln(y^2+z^2) = 0$$

Suppose $f(0,\frac{1}{\sqrt{2e}}) = \frac{1}{\sqrt{2e}}$. Find $f_y(0,\frac{1}{\sqrt{2e}})$.

Solution. Let $F(x,y,z) = \cos x + \ln(y^2+z^2)$.

So $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$, i.e. $f_y(x,y) = -\frac{F_y(x,y,z)}{F_z(x,y,z)}$.

$$F_y = \frac{2y}{y^2+z^2} \quad \text{and} \quad F_z = \frac{2z}{y^2+z^2}.$$

Hence
$$f_y(0,\tfrac{1}{\sqrt{2e}}) = -\frac{F_y(0,\frac{1}{\sqrt{2e}},\frac{1}{\sqrt{2e}})}{F_z(0,\frac{1}{\sqrt{2e}},\frac{1}{\sqrt{2e}})} = -1.$$

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