Summary of Week 5's lectures

The concept of limit for two or more variable functions is similar to the single variable case. Namely we say

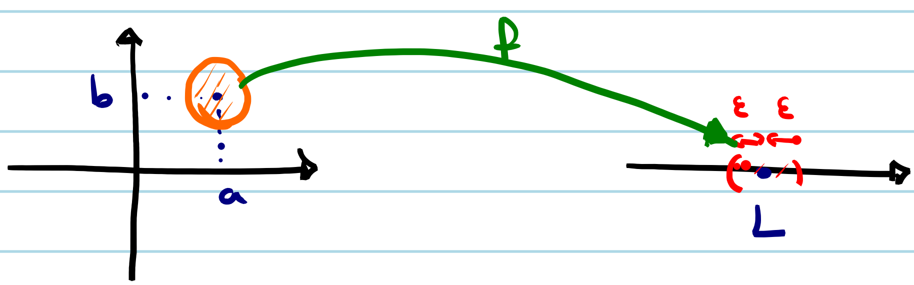
$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if someone demands for an ε-approximation of L via values of f, you would be able to say that it is enough to get a δ-approximation of (a,b).

Mathematically we write

For any $\varepsilon>0$, there is $\delta>0$ such that

$$\|(x,y)-(a,b)\|<\delta \Rightarrow |f(x,y)-L|<\varepsilon.$$



In this course, you are expected to be able

① Compute very easy limits:

When f is the composite of well-know

nice functions and no surprises happens when you evaluate $f(a,b)$.

② Compute limite of functions of the form

$$f(x,y) = g(x) \cdot h(y).$$

③ Use the following to possibly get a simpler limit:

$$\lim_{(x,y)\to(a,b)} g(f(x,y)) = L \quad \text{if}$$

$$\lim_{t\to \ell} g(t) = L \quad \text{and} \quad \lim_{(x,y)\to(a,b)} f(x,y) = L.$$

Exp. $$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = 1$$

Since $$\lim_{(x,y)\to(0,0)} x^2+y^2 = 0 \quad \text{and}$$

$$\lim_{t\to 0} \frac{\sin t}{t} = \lim_{t\to 0} \frac{\cos t}{1} = 1.$$

④ Use limits along lines to show

$$\lim_{(x,y)\to(a,b)} f(x,y)$$

does NOT exist:

If we compute $\lim\limits_{x \to 0} f(x, cx)$

and the answer depends on $\underline{\underline{c}}$, then

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

does not exist.

Remark. For a limit when $(x,y) \to (a,b) \neq (0,0)$ you have to use the lines

$$y = c(x-a) + b.$$

So you have to compute

$$\lim_{x \to a} f(x, c(x-a)+b)$$

and see if it depends on c or not.

5 Sometimes limit along all the lines are the same. It might be helpful to try other curves. For instance, if $f(x,y)$ is ratio of two polynomials, for $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ it is helpful to consider

$y = x^c$ or $x = y^c$ so that you get the same <u>deg</u> in the numerator and the denom.

<u>Exp.</u> Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6+y^2}$ does NOT exist.

<u>Solution</u>. <u>Along $y = cx$</u>

$$\lim_{x\to 0} \frac{x^3(cx)}{x^6+c^2x^2} = \lim_{x\to 0} \frac{c\,x^2}{x^4+c^2} = 0 .$$

<u>along $y = x^3$</u>

$$\lim_{x\to 0} \frac{(x^3)(x^3)}{x^6+x^6} = \frac{1}{2}$$

Since $0 \neq 1/2$, the two-variable limit does NOT exist.

⑥ use polar coordinates and squeeze theorem to compute a limit or to show it does NOT exist.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{\substack{r\to 0 \\ \text{uniform on } \theta}} f(r\cos\theta, r\sin\theta)$$

First Compute $f(r\cos\theta, r\sin\theta)$

Second Find $g_1(r)$ and $g_2(r)$ such that

$\boxed{A}$ $\lim_{r\to 0} g_1(r) = \lim_{r\to 0} g_2(r) = L$

$\boxed{B}$ $g_1(r) \le f(r\cos\theta, r\sin\theta) \le g_2(r)$

Third Use Squeeze Theorem and conclude that $\lim_{(x,y)\to(0,0)} f(x,y) = L$.

Exp. Let $a \ge 0$, $b \ge 0$. Show that

$$\lim_{(x,y)\to(0,0)} \frac{x^a y^b}{x^2+y^2} \begin{cases} \text{does NOT exist} & \text{if } a+b \le 2 \\ = 0 & \text{if } a+b > 2. \end{cases}$$

Solution.

$$\lim_{(x,y)\to(0,0)} \frac{x^a y^b}{x^2+y^2} = \lim_{\substack{r\to 0 \\ \text{uniform} \\ \text{on } \theta}} \frac{r^{a+b} \cdot \cos^a\theta \cdot \sin^b\theta}{r^2}$$

$$= \lim_{\substack{r\to 0 \\ \text{uniform on } \theta}} r^{a+b-2} \cdot \cos^a\theta \cdot \sin^b\theta$$

- If $a+b<2$, even for $\theta = \pi/4$

$$\lim_{r\to 0} \frac{\cos^a(\pi/4)\,\sin^b(\pi/4)}{r^{2-(a+b)}} = \infty .$$

- If $a+b=2$, then for a fixed θ we have

$$\lim_{r\to 0} \cos^a\theta\,\sin^b\theta = \cos^a\theta\,\sin^b\theta$$

which depends on $\underline{\underline{\theta}}$. So

$$\lim_{(x,y)\to(0,0)} \frac{x^a\,y^{2-a}}{x^2+y^2}$$

does not exist.

- If $a+b>2$, then

$$0 \le \left| r^{a+b-2} \cos^a\theta\,\sin^b\theta \right| \le r^{a+b-2}$$

and $\lim\limits_{r\to 0} 0 = \lim\limits_{r\to 0} r^{a+b-2} = 0$. Hence

by Squeeze theorem,

$$\lim_{(x,y)\to(0,0)} \frac{x^a\,y^b}{x^2+y^2} = 0 .$$

Exp. $\displaystyle\lim_{(x,y)\to(0,0)} \sin(xy)\cos\left(\frac{1}{x^2+y^2}\right)$

Solution. $\quad 0 \le |f(x,y)| \le |\sin(xy)| \longrightarrow 0$

By the squeeze theorem, we have

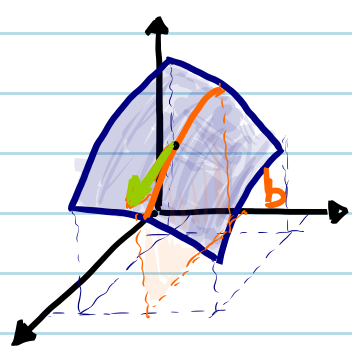
$$\lim_{(x,y)\to(0,0)} f(x,y) = 0. \qquad \blacksquare$$

Partial derivates

In order to understand the behavior of $f(x,y)$ around the point (a,b), first we focus on one variable at a time. We fix $\underline{y=b}$ and let x vary to find the rate of change of f with respect to x, i.e. find the derivative of $f(x,b)$.

Geometrically we look at the curve of intersection of $z=f(x,y)$ and $y=b$, and parametrize it using x as a parameter:

$$\vec{r}(x) = \langle x, b, f(x,b)\rangle$$

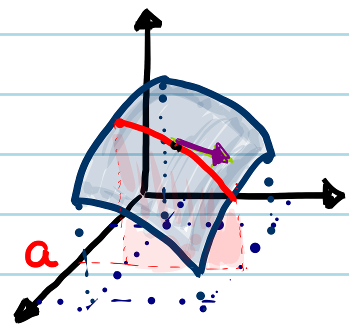


Now we would like to find the vector parametrization of its tangent line at $x=a$: $\vec{r}(a)+t\,\vec{r}\,'(a)$

$\vec{r}\,'(x) = \langle 1, 0, \underline{\quad} \rangle$ $\longrightarrow$ is the partial derivative of f with respect to x. It is denoted by $\frac{\partial f}{\partial x}$ or f_x.

So the tangent line is $\langle a, b, f(a,b)\rangle + t\langle 1, 0, f_x(a,b)\rangle$.

A vector parametrization of the curve of intersection of



$$\begin{cases} z = f(x,y) \\ x = a \end{cases}$$

is $\langle a, y, f(a,y)\rangle$. So its tangent line at $y=b$ is $\langle 0, 1, f_y(a,b)\rangle$.

<u>Partial derivative from computational point of view</u>

<u>Exp.</u> $z = x^2 + xy - y^2 \Rightarrow f_x = 2x + y$

$$f_y = x - 2y$$

<u>Exp.</u> $z = \sin\left(\frac{x}{y}\right) \Rightarrow f_x = \frac{1}{y}\cos\left(\frac{x}{y}\right)$ [Chain rule]

$$f_y = \frac{-x}{y^2}\cos\left(\frac{x}{y}\right)$$

Higher order partial derivatives

f_x is a two (or more) variable function. So we can ask about its partial derivatives:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) =: \frac{\partial^2 f}{\partial x^2} = (f_x)_x =: f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) =: \frac{\partial^2 f}{\partial y \partial x} = (f_x)_y =: f_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) =: \frac{\partial^2 f}{\partial x \partial y} = (f_y)_x =: f_{yx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) =: \frac{\partial^2 f}{\partial y^2} = (f_y)_y =: f_{yy}$$

Notice: $\dfrac{\partial^2 f}{\partial x \partial y} = f_{yx}$

Exp. $f(x,y) = \ln(x^2+y^2)$. Find all the second partial derivatives.

Solution. $f_x = \dfrac{2x}{x^2+y^2}$ (chain rule)

$f_y = \dfrac{2y}{x^2+y^2}$

$$f_{xx} = (f_x)_x = \frac{(2)(x^2+y^2) - (2x)(2x)}{(x^2+y^2)^2} = 2\,\frac{y^2-x^2}{(x^2+y^2)^2}$$

$$f_{xy} = (f_x)_y = \frac{(0)(x^2+y^2)-(2x)(2y)}{(x^2+y^2)^2} = \frac{-4xy}{(x^2+y^2)^2}$$

$$f_{yx} = (f_y)_x = \frac{(0)(x^2+y^2)-(2y)(2x)}{(x^2+y^2)^2} = \frac{-4xy}{(x^2+y^2)^2}$$

$$f_{yy} = (f_y)_y = \frac{(2)(x^2+y^2)-(2y)(2x)}{(x^2+y^2)^2} = 2\,\frac{x^2-y^2}{(x^2+y^2)^2}.$$

We observe that the second partial derivatives of the above function satisfy:

① $f_{xy} = f_{yx}$ ② $f_{xx} + f_{yy} = 0$

The first property is fairly general. The second property, however, holds only for a special class of functions called HARMONIC FUNCTIONS.

<u>Theorem</u> If f_{xy} and f_{yx} are continuous in a disk around (a,b), then $f_{xy}(a,b) = f_{yx}(a,b)$.

<u>Exp</u>. [Changing the order might help]

Find f_{xy} where $f(x,y) = e^{e^{(\ln x)^2}} + xy$

<u>Solution</u>. It is clear that f_{xy} and f_{yx} are

continuous in the domain of f. So

$$f_{xy} = f_{yx} = (f_y)_x \underset{\uparrow}{=} 1$$

$$\boxed{f_y = x}$$

Tangent plane.

Since graph $z = f(x,y)$ of f is a surface, it is more interesting to find a tangent plane rather than tangent lines.

If there is a tangent plane then it should contain the tangent lines that we found above:

$$\langle a, b, f(a,b) \rangle + t\langle 1, 0, f_x(a,b) \rangle \quad \text{and}$$

$$\langle a, b, f(a,b) \rangle + t\langle 0, 1, f_y(a,b) \rangle.$$

To find the equation of this plane we need to compute its normal vector:

$$\vec{n} = \langle 1, 0, f_x(a,b) \rangle \times \langle 0, 1, f_y(a,b) \rangle$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x(a,b) \\ 0 & 1 & f_y(a,b) \end{vmatrix} = \langle -f_x(a,b), -f_y(a,b), 1 \rangle.$$

So the equation of the possible tangent plane is

$$\vec{n} \cdot \langle x, y, z \rangle = \vec{n} \cdot \underbrace{\langle a, b, f(a,b) \rangle}_{\text{a point in the tangent plane}}$$

$$\Rightarrow \vec{n} \cdot \langle x-a, y-b, z-f(a,b) \rangle = 0$$

$$\Rightarrow -f_x(a,b)(x-a) - f_y(a,b)(y-b) + z - f(a,b) = 0$$

$$\Rightarrow z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

So the tangent plane at the point $(a, b, f(a,b))$ is the graph of the function

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Hence $L(x,y)$ is a good approximation of $f(x,y)$ if f has a tangent plane.