Summary the second week's lectures.

Dot Product

$$\left.\begin{array}{l}\vec{v} = \langle a_1, b_1, c_1\rangle \\ \vec{w} = \langle a_2, b_2, c_2\rangle\end{array}\right\} \Rightarrow \vec{v}.\vec{w} = a_1a_2 + b_1b_2 + c_1c_2 .$$

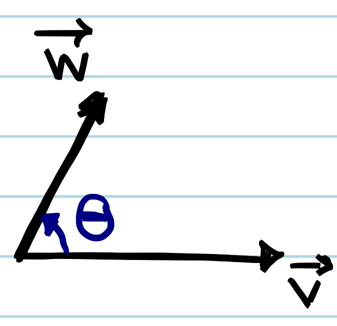
Warning. Dot product is a number. It is NOT a vector.

Algebraic Properties

1. $(c\vec{v})\cdot\vec{w} = c\,\vec{v}.\vec{w} = \vec{v}.(c\vec{w})$.
2. $\vec{v}.(\vec{w_1} + \vec{w_2}) = \vec{v}.\vec{w_1} + \vec{v}.\vec{w_2}$.
3. $(\vec{v_1} + \vec{v_2})\cdot\vec{w} = \vec{v_1}.\vec{w} + \vec{v_2}\cdot\vec{w}$.
4. $\vec{v}.\vec{w} = \vec{w}.\vec{v}$.

Geometric Properties

1. $\vec{v}.\vec{v} = \|\vec{v}\|^2$.
2. $\vec{v}.\vec{w} = \|\vec{v}\|\,\|\vec{w}\| \cos\theta$



Some of the examples and applications:

<u>Exp.</u> [The angle between two vectors is acute if and only if their dot product is positive.]

Is the angle between $\langle 1,2,0\rangle$ and $\langle -2,3,1\rangle$ acute ?

<u>Solution</u>. Since $\langle 1,2,0\rangle \cdot \langle -2,3,1\rangle =$

$$(1)(-2)+(2)(3)+(0)(1) = 4 > 0 ,$$

the angle between these vectors is acute.

<u>Exp.</u> [Two vectors $\vec{v}$ and $\vec{w}$ are orthogonal if and only if $\vec{v}.\vec{w}=0$.]

Find all values of x such that $\vec{v} = \langle x, -x+1, 1\rangle$ and $\vec{w} = \langle 1, x+1, x\rangle$ are perpendicular.

<u>Solution</u>. $\vec{v} \perp \vec{w}$ if and only if $\vec{v}.\vec{w} = 0$.

So $\langle x, -x+1, 1\rangle \cdot \langle 1, x+1, x\rangle =$

$$(x)(1) + (-x+1)(x+1) + (1)(x) =$$

$$x + (-x^2+1) + x =$$

$$-x^2 + 2x + 1 = 0$$

Thus $x^2 - 2x + 1 = 2$

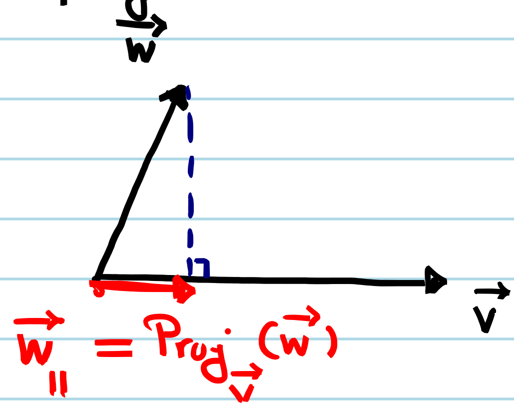
Therefore $(x-1)^2 = 2$

which implies $x = 1+\sqrt{2}$ or $x = 1-\sqrt{2}$. $\blacksquare$

Orthogonal Projection

$\text{Proj}_{\vec{v}}(\vec{w})$: the orthogonal projection of $\vec{w}$ along $\vec{v}$.

$$\text{Proj}_{\vec{v}}(\vec{w}) = \frac{\vec{v}\cdot\vec{w}}{\vec{v}\cdot\vec{v}}\,\vec{v}.$$



Remark. [Geometric interpretation of dot product]

$$|\vec{v}\cdot\vec{w}| = \|\text{Proj}_{\vec{v}}(\vec{w})\|\,\|\vec{v}\|$$
$$= \left(\begin{array}{c}\text{length of the projection}\\ \text{of } \vec{w} \text{ along } \vec{v}\end{array}\right)(\text{length of } \vec{v}).$$

- $\text{Proj}_{\vec{v}}(\vec{w}) = \left(\begin{array}{c}\text{The component of } \vec{w}\\ \text{along } \vec{v}\end{array}\right)\vec{e}_{\vec{v}}$,

where $\vec{e}_{\vec{v}} = \dfrac{\vec{v}}{\|\vec{v}\|}$ is the unit vector in the direction of $\vec{v}$.

• The component of $\vec{w}$ along $\vec{v} = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|}$.

Exp. [How to use length of vectors to compute dot product.]

Assume $\|\vec{v}\| = 2$, $\|\vec{w}\| = 3$ and $\|2\vec{v}+\vec{w}\| = 6$.

(a) Find $\vec{v}\cdot\vec{w}$.

(b) Find $\|\vec{v}+3\vec{w}\|$.

Solution. (a) $\|2\vec{v}+\vec{w}\|^2 = 4\,\vec{v}\cdot\vec{v} + 4\,\vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{w}$

$$= 4\|\vec{v}\|^2 + 4\,\vec{v}\cdot\vec{w} + \|\vec{w}\|^2$$

$\Rightarrow \quad 36 = (4)(4) + 4\,\vec{v}\cdot\vec{w} + 9$

$\Rightarrow \quad 4\,\vec{v}\cdot\vec{w} = 36 - 16 - 9 = 11$

$\Rightarrow \quad \vec{v}\cdot\vec{w} = 11/4$.

(b) $\|\vec{v}+3\vec{w}\|^2 = \vec{v}\cdot\vec{v} + 6\,\vec{v}\cdot\vec{w} + 9\,\vec{w}\cdot\vec{w}$

$$= \|\vec{v}\|^2 + 6\,\vec{v}\cdot\vec{w} + 9\,\|\vec{w}\|^2$$

$$= 4 + (6)(11/4) + (9)(9)$$

$$= 4 + \frac{33}{2} + 81 = 101.5$$

$\Rightarrow \quad \|\vec{v}+3\vec{w}\| = \sqrt{101.5}$.

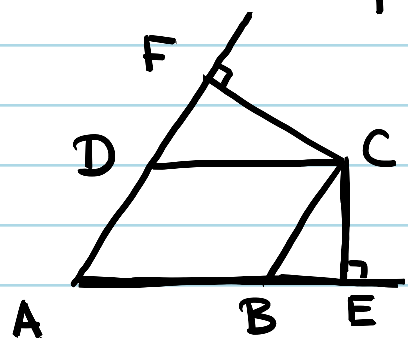
Advance Example [NOT needed for the exam.]

In the following figure ABCD is a parallelogram.

Show that

$$AC^2 = AB \cdot AE + AD \cdot AF$$

(These are NOT dot products.) [Come to my office hour for its solution]



Determinant of 2x2 and 3x3 matrices

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Alternative way ①

$$\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix}$$

$-$: $a_3 b_2 c_1$, $a_1 b_3 c_2$, $a_2 b_1 c_3$

$+$: $a_1 b_2 c_3$, $a_2 b_3 c_1$, $a_3 b_1 c_2$

Alternative way ②

$$\begin{matrix} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{matrix}$$

$$-a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3 + a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1$$

Cross Product of $\vec{v}$ and $\vec{w}$

$$\left.\begin{array}{l}\vec{v} = \langle a_1, b_1, c_1 \rangle \\ \vec{w} = \langle a_2, b_2, c_2 \rangle\end{array}\right\} \Rightarrow \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}.$$

Algebraic Properties

① $(c\vec{v}) \times \vec{w} = c\, \vec{v} \times \vec{w} = \vec{v} \times (c\vec{w})$.

② $\vec{v} \times (\vec{w}_1 + \vec{w}_2) = \vec{v} \times \vec{w}_1 + \vec{v} \times \vec{w}_2$.

③ $(\vec{v}_1 + \vec{v}_2) \times \vec{w} = \vec{v}_1 \times \vec{w} + \vec{v}_2 \times \vec{w}$.

④ $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$.

Geometric Properties

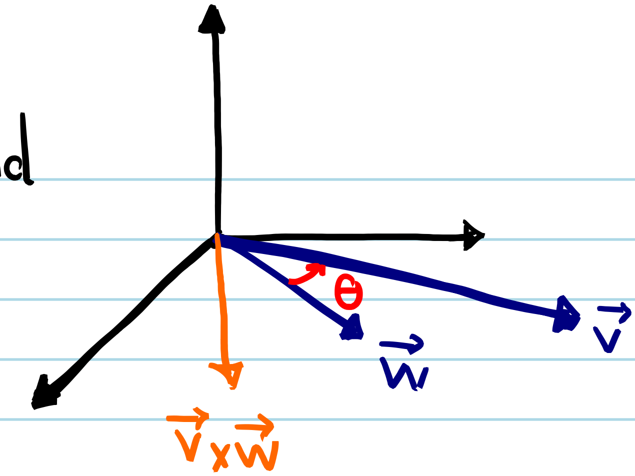
① $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \|\vec{w}\| \sin\theta$, where $0 \le \theta \le \pi$ is the angle between $\vec{v}$ and $\vec{w}$.

② $\vec{v} \times \vec{w} \perp \vec{v}$ and $\vec{v} \times \vec{w} \perp \vec{w}$.

③ $\vec{v}$, $\vec{w}$ and $\vec{v} \times \vec{w}$ obey the right hand rule.

<u>Exp.</u> In the following figure the vectors $\vec{v}$ and $\vec{w}$ are parallel to the xy-plane. Find $\vec{v} \times \vec{w}$

if $\|\vec{v}\| = 2$, $\|\vec{w}\| = 1$ and $\theta = \pi/6$.



<u>Solution</u>. We know

$$\|\vec{v}\times\vec{w}\| = \|\vec{v}\|\,\|\vec{w}\|\,\sin(\theta)$$
$$= (2)(1)(\sin(\pi/6))$$
$$= (2)(1)(1/2)$$
$$= 1.$$

We also know $\vec{v}\times\vec{w}$ is perpendicular to $\vec{v}$ and $\vec{w}$. So it is perpendicular to the xy-plane. Hence $\vec{v}\times\vec{w} = c\vec{k}$ for some number c. Since $\|\vec{v}\times\vec{w}\| = 1$, $c = \pm 1$.

At last we know that $\vec{v}$, $\vec{w}$ and $\vec{v}\times\vec{w}$ obey the right hand rule, so $\vec{v}\times\vec{w}$ is in the direction of negative z-axis. Hence

$$\vec{v}\times\vec{w} = \langle 0, 0, -1\rangle.$$

Exp. [Area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$ is equal to $\|\vec{v}\times\vec{w}\|$.]

Find the area of the parallelogram spanned by

$$\vec{v}=\langle 1,2,0\rangle \quad \text{and} \quad \vec{w}=\langle -2,1,1\rangle .$$

Solution. Area $=\|\vec{v}\times\vec{w}\|$.

$$\vec{v}\times\vec{w}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ -2 & 1 & 1 \end{vmatrix}=\left\langle \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} \right\rangle$$

$$=\langle 2,-1,5\rangle$$

$$\Rightarrow \text{Area}=\sqrt{(2)^2+(-1)^2+(5)^2}=\sqrt{4+1+25}=\sqrt{30}.$$

Exp. [Volume of the parallelpiped spanned by $\vec{u}$, $\vec{v}$ and $\vec{w}$ is $|\vec{u}\cdot(\vec{v}\times\vec{w})|=\left|\det\begin{bmatrix}\vec{u}\\ \vec{v}\\ \vec{w}\end{bmatrix}\right|$.]

Find the volume of the parallelpiped spanned by

$\vec{u}=\langle 1,2,0\rangle$, $\vec{v}=\langle -1,2,1\rangle$ and $\vec{w}=\langle 2,1,-1\rangle$.

Solution.

$$\text{Volume}=\left|\det\begin{bmatrix} 1 & 2 & 0 \\ -1 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix}\right|$$

$$\begin{array}{ccc|cc} 1 & 2 & 0 & 1 & 2 \\ -1 & 2 & 1 & -1 & 2 \\ 2 & 1 & -1 & 2 & 1 \end{array}$$

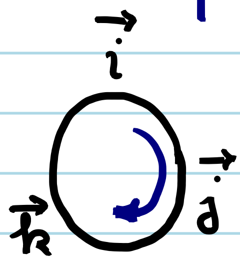
$$\underbrace{0 \quad 1 \quad 2}_{\ominus} \qquad \underbrace{-2 \quad 4 \quad 0}_{\oplus}$$

$$\Rightarrow \det\begin{bmatrix} 1 & 2 & 0 \\ -1 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix}$$

$$= -2+4-1-2$$

$$= -1.$$

So volume $= |-1| = 1$.

Exp. [Use  and distribution to compute cross product.]

Find $(2\vec{i}+3\vec{j})\times(\vec{j}-4\vec{k})$.

Solution. $(2\vec{i}+3\vec{j})\times(\vec{j}-4\vec{k})$

$$= 2\underbrace{\vec{i}\times\vec{j}}_{\vec{k}} - 8\underbrace{\vec{i}\times\vec{k}}_{-\vec{j}} + 3\underbrace{\vec{j}\times\vec{j}}_{\vec{0}} - 12\underbrace{\vec{j}\times\vec{k}}_{\vec{i}}$$

$$= -12\vec{i} + 8\vec{j} + 2\vec{k}$$

$$= \langle -12, 8, 2\rangle.$$