Math 21C Final Exam Lecture B Agler Winter 02

1. Use the cross product to calculate the area of the triangle with vertices $(1, 1, 1), (2, 3, 2)$, and $(3, -1, 4)$.

2. At what point do the curves $\vec{r_1}\,(t) = < t, t^2, t^3 >$ and $\vec{r_2}\,(t) = < 1 + t, 4t, 8t^2 >$ intersect? Find their angle of intersection to the nearest degree.

3. Find an equation for the planes consisting of all points that are equidistant from the points $(1, 2, 3)$ and $(-1, 1, -1)$.

4. For $0 \le t \le 1$ a particle moves with position vector given by $\vec{r}\,(t) = 2t^{3/2}\,\vec{i} + cos\,2t\,\vec{j} + \sin 2t\,\vec{k}$. Find the initial speed of the particle and the total distance it travels.

5. Find the points on the ellipsoid $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$ where the tangent plane is parallel to the plane $z = x + y$.

6. Find and classify the critical points of $f(x, y) = x^4 - 8xy + 2y^2 - 3$.

7. A cardboard box without a lid is to have a volume of 32,000 cm^3. Find the dimensions that minimize the amount of cardboard used.

8. Find the volume of the solid bounded by the paraboloid $z = 10 - 3x^2 - 3y^2$ and the plane $z = 4$.

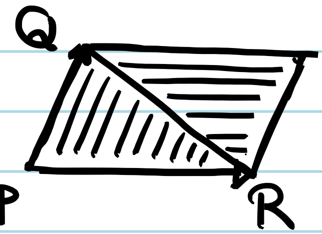
9. Find the area of the part of the surface $z = x + y^2$ that lies above the triangle with vertices $(0, 0), (1, 1)$, and $(0, 1)$.

10. Evaluate $\iiint_E y\,dV$ where E is the solid tetrahedron with vertices $(0, 0, 0), (1, 0, 0), (0, 1, 0)$ and $(0, 0, 2)$.

NOT part of our exam.

Use the cross product to calculate the area of the triangle with vertices (1, 1, 1), (2, 3, 2), and (3, -1, 4).

Area of the parallelogram spanned by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is $\|\overrightarrow{PQ} \times \overrightarrow{PR}\|$. So area of the triangle PQR is $\frac{1}{2}\|\overrightarrow{PQ} \times \overrightarrow{PR}\|$.



Let $P = (1,1,1)$,

$Q = (2,3,2)$, and $R = (3,-1,4)$. So

$\overrightarrow{PQ} = \langle 1,2,1 \rangle$ and $\overrightarrow{PR} = \langle 2,-2,3 \rangle$.

$$\Rightarrow \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ 2 & -2 & 3 \end{vmatrix}$$

$$= \left\langle \begin{vmatrix} 2 & 1 \\ -2 & 3 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} \right\rangle$$

$$= \langle 6+2, -(3-2), -2-4 \rangle$$

$$= \langle 8, -1, -6 \rangle$$

$$\Rightarrow \|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \sqrt{64+1+36} = \sqrt{101}$$

$\Rightarrow$ area of the triangle $PQR = \dfrac{\sqrt{101}}{2}$.

2. $\vec{r}_1(t) = \langle t, t^2, t^3 \rangle$ and $\vec{r}_2(t) = \langle 1+t, 4t, 8t^2 \rangle$

$$\begin{cases} t = 1+s \\ t^2 = 4s \\ t^3 = 8s^2 \end{cases} \xRightarrow[\text{equations}]{\text{first two}} (1+s)^2 = 4s$$

$$\Rightarrow s^2 + 2s + 1 = 4s$$

$$\Rightarrow s^2 - 2s + 1 = 0$$

$$\Rightarrow (s-1)^2 = 0$$

$$\Rightarrow s = 1$$

$$\Rightarrow t = 2$$

$\Rightarrow \vec{r}_1(2) = \langle 2, 4, 8 \rangle = \vec{r}_2(1)$ is the only intersection point of these curves.

Angle between curves at P_0 is the angle between their tangent vectors $\Rightarrow$

$\vec{r}_1{}'(t) = \langle 1, 2t, 3t^2 \rangle \Rightarrow \vec{r}_1{}'(2) = \langle 1, 4, 12 \rangle$

$\vec{r}_2{}'(s) = \langle 1, 4, 16s \rangle \Rightarrow \vec{r}_2{}'(1) = \langle 1, 4, 16 \rangle$

$$\Rightarrow \cos\theta = \frac{\vec{r}_1{}'(2) \cdot \vec{r}_2{}'(1)}{\|\vec{r}_1{}'(2)\| \, \|\vec{r}_2{}'(1)\|}$$

$\|\vec{r}_1{}'(2)\| = \sqrt{1+16+144} = \sqrt{161}$,

$$\|\vec{r}_2\,'(1)\| = \sqrt{1+16+256} = \sqrt{273}$$

$$\vec{r}_1\,'(2)\cdot\vec{r}_2\,'(1) = 1+16+192 = 209$$

$$\Rightarrow \cos\theta = \frac{209}{\sqrt{161}\cdot\sqrt{273}}$$

I used calculator to get $\theta \approx 4.51^\circ$. In our exam, you are NOT allowed to use calculator.

3. <u>Solution 1</u>.

Plane passes through the middle point and perpendicular to the segment. So

$\vec{n} = \langle 1+1, 2-1, 3+1\rangle = \langle 2,1,4\rangle$

is a normal vector, and it passes through

$$\left(\frac{1-1}{2}, \frac{2+1}{2}, \frac{3-1}{2}\right) = \left(0, \frac{3}{2}, 1\right)$$

$$\Rightarrow 2x+y+4z = (2)(0)+\frac{3}{2}+(4)(1)$$

$$\Rightarrow 2x+y+4z = \frac{11}{2}.$$

Solution 2. $\sqrt{(x-1)^2+(y-2)^2+(z-3)^2} = \sqrt{(x+1)^2+(y-1)^2+(z+1)^2}$

$$\Rightarrow \cancel{x^2} - 2x + \cancel{1} + \cancel{y^2} - 4y + 4 + \cancel{z^2} - 6z + 9 =$$

$$\cancel{x^2} + 2x + \cancel{1} + \cancel{y^2} - 2y + 1 + \cancel{z^2} + 2z + 1$$

$$\Rightarrow 4x + 2y + 8z = 11$$

4. $\vec{r}(t) = \langle 2t^{3/2}, \cos(2t), \sin(2t) \rangle$ for $0 \leq t \leq 1$

$$\Rightarrow \vec{r}\,'(t) = \langle 3t^{1/2}, -2\sin(2t), 2\cos(2t) \rangle$$

$$\Rightarrow s(t) = \|\vec{r}\,'(t)\| = \sqrt{(3t^{1/2})^2 + 4\sin^2(2t) + 4\cos^2(2t)}$$

$$= \sqrt{9t+4}.$$

$\Rightarrow$ initial speed $= s(0) = \sqrt{4} = 2$.

Total distance $= \int_0^1 s(t)\,dt$

$$= \int_0^1 \sqrt{9t+4}\,dt$$

$$\boxed{u = 9t+4 \Rightarrow du = 9\,dt}$$

$$= \int_4^{13} \sqrt{u}\,\frac{du}{9} = \frac{1}{9} \cdot \frac{2}{3} \cdot u^{\frac{3}{2}} \Big|_4^{13}$$

$$= \frac{2}{27}\left(13\sqrt{13} - 8\right).$$

5. $\nabla f(P_0)$ is a normal vector of the tangent plane of the level surface $f(x,y,z) = \text{const.}$ at P_0.

Two planes are parallel if and only if their normal vectors are parallel.

So a normal vector of the tangent plane of $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$ at (x,y,z) is

$$\langle 2x, 2\,y/4, 2\,z/9 \rangle,$$

which is supposed to be parallel to $\langle 1, 1, -1 \rangle$ (a normal vector of $x+y-z=0$).

$$\Longrightarrow \begin{cases} \langle 2x, y/2, 2\,z/9 \rangle = c\langle 1,1,-1 \rangle \\ \quad \text{for some } c \\ x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1 \end{cases}$$

$$\begin{cases} c = 2x \\ c = y/2 \\ -c = 2\,z/9 \end{cases} \Rightarrow y = 4x \text{ and } z = -9x$$

$$\Rightarrow 1 = x^2 + \frac{y^2}{4} + \frac{z^2}{9} = x^2 + 4x^2 + 9x^2$$

$$\Rightarrow 1 = 14x^2 \Rightarrow x = \pm\frac{1}{\sqrt{14}} = \pm\frac{\sqrt{14}}{14}$$

So the desired points are

$$\pm\frac{\sqrt{14}}{14}(1,4,-9).$$

6. $\nabla f = \langle 4x^3 - 8y, -8x + 4y\rangle = \langle 0,0\rangle$

$$\begin{cases} 4x^3 - 8y = 0 \\ -8x + 4y = 0 \end{cases} \Rightarrow \begin{cases} x^3 = 2y \\ 2x = y \end{cases} \Rightarrow x^3 = 4x$$

$$\Rightarrow x = 0 \text{ or } x = \pm 2$$

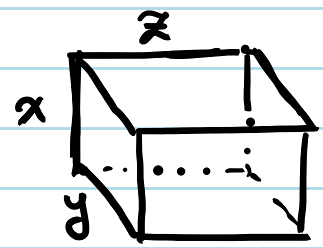
$\Rightarrow$ critical points are $(0,0), (2,4), (-2,-4)$.

Critical pts	$D = f_{xx} \cdot f_{yy} - (f_{xy})^2$	f_{xx}	result
$(0,0)$	$-$		Saddle pt
$(2,4)$	$+$	$+$	local min
$(-2,-4)$	$+$	$+$	local min

$$\left.\begin{array}{l} f_{xx} = 12x^2 \\ f_{yy} = 4 \\ f_{xy} = -8 \end{array}\right\} \Rightarrow D = 48x^2 - 64$$

$$= 16(3x^2 - 4)$$

7. $xyz = 32\,000$



Find the min. of

$2xz + 2xy + yz$ (no lid)

We can use Lagrange multiplier:

Find the min. $f(x,y,z) = 2xz + 2xy + yz$

under the constraint $g(x,y,z) = xyz = 32\,000$.

$$\begin{cases} \nabla f = c\,\nabla g \\ g = 32000 \end{cases} \Rightarrow$$

$$\begin{cases} \langle 2z + 2y,\ 2x + z,\ 2x + y \rangle = c \langle yz, xz, xy \rangle \\ xyz = 32000 \end{cases}$$

Since $xyz \neq 0$, none of them is zero.

$$\Rightarrow c = \frac{2z + 2y}{yz} = \frac{2x + z}{xz} = \frac{2x + y}{xy}$$

$$\Rightarrow \begin{cases} 2xz^2 + 2x\cancel{y}z = 2\cancel{x}yz + yz^2 \\ 2x\cancel{y}z + 2xy^2 = 2\cancel{x}yz + y^2 z \end{cases}$$

$$\Rightarrow \begin{cases} 2x = y \\ 2x = z \\ xyz = 32\,000 \end{cases} \Rightarrow 4x^3 = 32\,000$$

$$\Rightarrow x^3 = 8000$$

$$\Rightarrow x = 20$$

$$\Rightarrow y = z = 40.$$

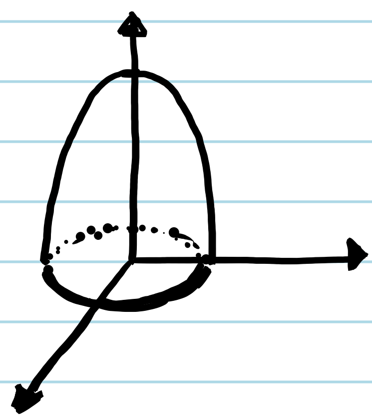
8. Paraboloid $z = 10 - 3x^2 - 3y^2$ and $z = 4$.

First we translate the solid so it becomes the solid between $z = 6 - 3x^2 - 3y^2$ and $z = 0$.

The intersection is

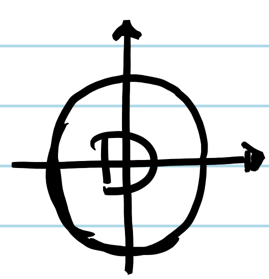
$$6 - 3x^2 - 3y^2 = 0$$

in the xy-plane.



$\Rightarrow x^2 + y^2 = 2$ in the xy-plane

$$\text{vol} = \iint_D 6 - 3x^2 - 3y^2 \, dA$$



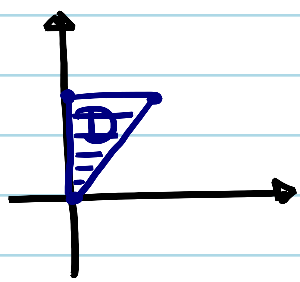
$$= \int_0^{2\pi} \int_0^{\sqrt{2}} (6 - 3r^2)\, r \, dr \, d\theta$$

$$A(\theta) = \int_0^{\sqrt{2}} 6r - 3r^3 \, dr = \left(3r^2 - \frac{3}{4} r^4\right)\Big|_0^{\sqrt{2}}$$

$$= 6 - 3 = 3$$

$$\Rightarrow \text{vol} = \int_0^{2\pi} 3 \, d\theta = 6\pi.$$

9. $\text{vol.} = \iint_D x + y^2 \, dA$



$0 \le y \le 1$ and $0 \le x \le y$

$$\Rightarrow \text{vol} = \int_0^1 \int_0^y x + y^2 \, dx \, dy$$

$$A(y) = \int_0^y x + y^2 \, dx = \left(\frac{x^2}{2} + y^2 x\right)\Big|_{x=0}^{x=y}$$

$$= \frac{y^2}{2} + y^3$$

$$\Rightarrow \text{vol} = \int_0^1 \frac{y^2}{2} + y^3 \, dy = \left(\frac{y^3}{6} + \frac{y^4}{4}\right)\Big|_0^1 = \frac{1}{6} + \frac{1}{4}$$

$$\Rightarrow \text{vol} = \frac{5}{12}.$$