Print Name: ______________________ Student Number: _____________

Section Time: _____________

Math 20C.
Final Exam
December 8, 2005

Read each question carefully, and answer each question completely.
Show all of your work. No credit will be given for unsupported answers.
Write your solutions clearly and legibly. No credit will be given for illegible solutions.

1. (8 Pts.) Find the equation of the plane that contains both the point $(-1, 0, 1)$ and the line $x = t$, $y = -1 + 2t$, $z = 3t$.

It contains $\underbrace{(0,-1,0)}_{P}$, $\underbrace{(1,1,3)}_{Q}$ and $\underbrace{(-1,0,1)}_{R}$. So

$\vec{PQ} \times \vec{PR}$ is a normal vector.

$$\vec{PQ} \times \vec{PR} = \langle 1, 2, 3 \rangle \times \langle -1, 1, 1 \rangle = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ -1 & 1 & 1 \end{vmatrix}$$

$$= \left\langle \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} \right\rangle$$

$$= \langle -1, -4, 3 \rangle$$

$-x - 4y + 3z = 0 + 4 + 0$ (plug-in P)

$\Rightarrow$ $\boxed{-x - 4y + 3z = 4}$

#	Score
1	
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2. (8 Pts.) Find the values of the constants a and b such that the function

$$f(t,x) = \sin(x - at) + \cos(bx + t)$$

is solution of the wave equation $f_{tt} = 4f_{xx}$.

$$f_t = -a\cos(x-at) - \sin(bx+t)$$

$$\Rightarrow f_{tt} = -a^2 \sin(x-at) - \cos(bx+t)$$

$$f_x = \cos(x-at) - b\sin(bx+t)$$

$$\Rightarrow f_{xx} = -\sin(x-at) - b^2\cos(bx+t)$$

$$f_{tt} = 4f_{xx} \Leftrightarrow (a^2-4)\sin(x-at) = (4b^2-1)\cos(bx+t)$$

$\underline{x=t=0}$ $\Rightarrow$ LHS $=0$ $\Rightarrow 4b^2-1=0 \Rightarrow \boxed{b = \pm 1/2}$

$$\Rightarrow (a^2-4)\sin(x-at) = 0$$

$$\Rightarrow a^2-4=0$$

$$\Rightarrow \boxed{a = \pm 2}$$

3. (6 Pts.) Consider the function $z(t) = f(x(t), y(t))$, where

$$f(x,y) = (2x+y^2)^{1/2}, \quad x(t) = e^{3t}, \quad y(t) = e^{-3t}.$$

Compute $\frac{d}{dt}z(t)$.

$$\frac{d}{dt} f(r(t)) = \nabla f(r(t)) \cdot \vec{r}\,'(t) .$$

$$\nabla f = \left\langle (2)\left(\tfrac{1}{2}\right)(2x+y^2)^{-1/2} , (2y)\left(\tfrac{1}{2}\right)(2x+y^2)^{-1/2} \right\rangle$$

$$= \frac{1}{\sqrt{2x+y^2}} \langle 1, y \rangle .$$

$$\nabla f(r(t)) = \frac{1}{\sqrt{2e^{3t}+e^{-6t}}} \langle 1, e^{-3t} \rangle .$$

$$\vec{r}\,'(t) = \langle 3e^{3t}, -3e^{-3t} \rangle$$

$$\frac{d}{dt} f(r(t)) = \frac{1}{\sqrt{2e^{3t}+e^{-6t}}} \left(3e^{3t} - 3e^{-6t} \right) .$$

4. (8 Pts.) The function $z(x, y)$ is defined implicitly by the equation $z^2xy = \cos(2x + z)$. Compute the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ as functions of x, y and z.

$$F(x,y,z) = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

$$F = z^2xy - \cos(2x+z) \Rightarrow$$

$$\begin{cases} F_x = z^2y + 2\sin(2x+z) \\ F_y = z^2x \\ F_z = 2xyz + \sin(2x+z) \end{cases}$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{z^2y + 2\sin(2x+z)}{2xyz + \sin(2x+z)}$$

$$\text{and} \quad \frac{\partial z}{\partial y} = -\frac{z^2x}{2xyz + \sin(2x+z)} .$$

5. (10 Pts.) Reparametrize the curve $\mathbf{r}(t) = (e^{2t}\cos(2t), 2, e^{2t}\sin(2t))$, with respect to the arc length measured from the point where $t = 0$ in the direction of increasing t.

Arc length param. is about the position of a particle which moves on the same path (curve), but with the constant speed 1.

Arc length para. is NOT part of our exam, but you might be asked to find the total distance travelled

· velocity: $r'(t) = \langle 2e^{2t}\cos(2t) - 2e^{2t}\sin(2t), 0, 2e^{2t}\sin(2t) + 2e^{2t}\cos(2t)\rangle$

· speed: $s(t) = \|r'(t)\|$

$$= 2e^{2t}\sqrt{(\cos(2t) - \sin(2t))^2 + (\sin(2t) + \cos(2t))^2}$$

$$= 2e^{2t}\sqrt{\cos^2 2t + \sin^2 2t - 2\cos 2t \sin 2t + \sin^2 2t + \cos^2 2t + 2\cos 2t \sin 2t}$$

$$= 2\sqrt{2}\, e^{2t}.$$

. Total distance travelled: $0 \le u \le t$

$$\int_0^t s(u)\,du = \int_0^t 2\sqrt{2}\, e^{2u}\,du = \sqrt{2}\, e^{2u}\Big|_0^t = \sqrt{2}(e^{2t} - 1).$$

This portion we did not cover

· $r_{arc}(s) = r(t_s)$ where t_s is the time where the total distance travelled is s:

$\sqrt{2}(e^{2t_s} - 1) = s \implies t_s = \frac{1}{2}\ln\left(\frac{s}{\sqrt{2}} + 1\right).$

$$\implies r_{arc}(s) = \left\langle \left(\frac{s+1}{\sqrt{2}}\right)\cos\left(\ln\left(\frac{s}{\sqrt{2}} + 1\right)\right), 2, \left(\frac{s+1}{\sqrt{2}}\right)\sin\left(\ln\left(\frac{s}{\sqrt{2}} + 1\right)\right)\right\rangle$$

6. (10 Pts.) Consider the function $f(x, y) = 8x^3 - 6xy + y^3$.

(a) Find the critical points of f.

(b) For each critical point determine whether it is a local maximum, local minimum, or a saddle point.

(a) $\nabla f = \langle 24x^2 - 6y, -6x + 3y^2 \rangle = \langle 0, 0 \rangle$

$\Rightarrow \begin{cases} 4x^2 = y \\ 2x = y^2 \end{cases} \Rightarrow 2x = (4x^2)^2 = 16x^4$

$\Rightarrow x = 0$ or $x^3 = \frac{1}{8}$

$\Rightarrow x = 0$ or $x = \frac{1}{2}$.

$\Rightarrow \begin{cases} y = 4x^2 = 0 \text{ if } x = 0 \\ y = 4x^2 = 1 \text{ if } x = \frac{1}{2} \end{cases} \Rightarrow$ critical pts are $(0,0)$ and $(\frac{1}{2}, 1)$.

(b) $f_{xx} = 48x$

$f_{xy} = -6$

$f_{yy} = 6y$

$D = f_{xx} f_{yy} - (f_{xy})^2 = (48)(6)xy - 6^2$

$= 36(8xy - 1)$

Critical pts	D	f_{xx}	result
$(0,0)$	-1	$\Rightarrow$	Saddle pt
$(\frac{1}{2}, 1)$	$+$	$+$	local min

7. (10 Pts.) Find the maximum and minimum values of the function $f(x,y) = x^2+2y^2-2x$ subject to the constraint $x^2+y^2=4$.

Since $x^2+y^2=4$ is bounded and closed, and f is continuous, f has a max. and a min on $x^2+y^2=4$.

So Lagrange multipliers method give us the result.

$$\begin{cases} \nabla f = c\nabla g \\ g = 4 \end{cases} \Rightarrow \begin{cases} \langle 2x-2, 4y\rangle = c\langle 2x, 2y\rangle \\ x^2+y^2=4 \end{cases}$$

(where $g(x,y) = x^2+y^2$)

$$\Rightarrow \begin{cases} 2x-2 = 2cx & \text{(i)} \\ 4y = 2cy & \text{(ii)} \\ x^2+y^2=4 & \text{(iii)} \end{cases}$$

(ii) $\Rightarrow$ $y=0$ or $c=2$.

If $y \neq 0$, then $c=2 \overset{\text{(i)}}{\Longrightarrow} x-1=2x \Rightarrow x=-1$

$\overset{\text{(iii)}}{\Longrightarrow} 1+y^2=4 \Rightarrow y=\pm\sqrt{3}$.

$\Rightarrow (-1,\sqrt{3})$ or $(-1,-\sqrt{3})$.

If $y=0$, then $\overset{\text{(iii)}}{\Longrightarrow} x^2=4 \Rightarrow x=\pm 2$

$\Rightarrow (2,0)$ or $(-2,0)$

relevant pts	$(-1,\sqrt{3})$	$(-1,-\sqrt{3})$	$(2,0)$	$(-2,0)$
value of f	9	9	0	8

maximum (at $(-1,\sqrt{3})$, $(-1,-\sqrt{3})$): 9

minimum (at $(2,0)$): 0

8. (a) (10 Pts.) Sketch the region of integration, D, whose area is given by the double integral

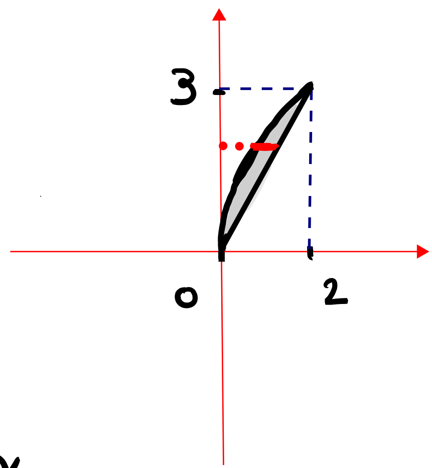
$$\int\int_D dA = \int_0^2 \int_{\frac{3}{2}x}^{3\sqrt{x/2}} dy\, dx.$$

(b) Compute the double integral given in (a).

(c) Change the order of integration in the integral given in (a). (You don't need to compute the integral again.)

$$A(x) = \int_{\frac{3}{2}x}^{3\sqrt{\frac{x}{2}}} dy$$

$$= y \Big|_{\frac{3}{2}x}^{3\sqrt{\frac{x}{2}}} = 3\sqrt{\frac{x}{2}} - \frac{3}{2}x$$



$$\Rightarrow \iint_D dA = \int_0^2 3\sqrt{\frac{x}{2}} - \frac{3}{2}x\, dx$$

$$= \left(\frac{3}{\sqrt{2}} \cdot \frac{2}{3} \cdot x^{\frac{3}{2}} - \frac{3}{2} \cdot \frac{1}{2} \cdot x^2\right)\Big|_0^2$$

$$= \sqrt{2} \cdot 2^{\frac{3}{2}} - \frac{3}{4} \cdot 2^2$$

$$= 4 - 3 = 1.$$

$$\iint_D dA = \int_0^3 \int_{2(y/3)^2}^{\frac{2}{3}y} dx\, dy$$

$$y = 3\sqrt{\frac{x}{2}} \Rightarrow \frac{y}{3} = \sqrt{\frac{x}{2}} \Rightarrow x = 2\left(\frac{y}{3}\right)^2$$

$$y = \frac{3}{2}x \Rightarrow x = \frac{2}{3}y$$

9. (10 Pts.) Compute the integral

$$I = \int_0^1 \int_0^y \int_0^{4-y^2} yz \, dx \, dz \, dy$$

NOT PART OF OUR EXAM

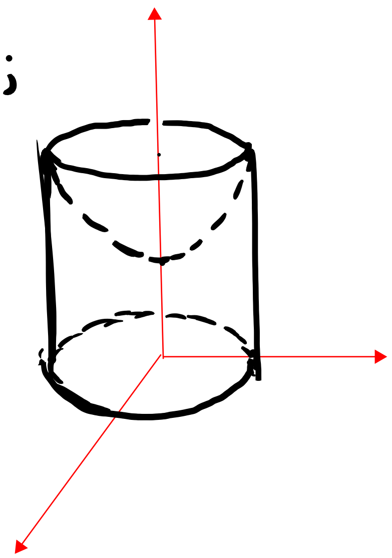
10. (10 Pts.) Consider the region of $D \subset I\!R^3$ given by

$$D = \{(x,y,z) \in I\!R^3 : x^2 + y^2 \leq 1, \quad 0 \leq z \leq 1 + x^2 + y^2\}.$$

(a) Sketch the region D.

(b) Compute the volume of that region.

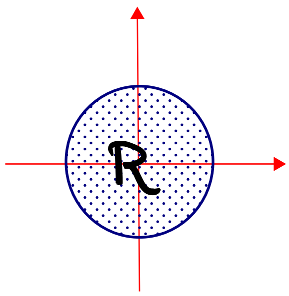
(a) Inside the cylinder $x^2+y^2 \leq 1$;
above the xy-plane;
under the graph $z = 1+x^2+y^2$;



$x^2+y^2=1$ and $z = 1+x^2+y^2$ intersect at $z = 1+1 = 2$ plane

(b) volume of the solid above the region R in the xy-plane and under $z = f(x,y)$ $= \iint_R f(x,y)\, dA$

So $\text{vol}(D) = \iint_R 1+x^2+y^2 \, dA$ where



$0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 1$ ← polar description of R

$$\Rightarrow \text{vol}(D) = \int_0^{2\pi}\int_0^1 (1+r^2)\, r\, dr\, d\theta = \int_0^{2\pi} \frac{3}{4}\, d\theta = \frac{3\pi}{2}$$

$$A(\theta) = \int_0^1 (1+r^2)\, r\, dr = \left(\frac{r^2}{2} + \frac{r^4}{4}\right)\Big|_0^1 = \frac{3}{4}$$