Print Name: ______________________ Section Number: ______

TA Name: ______________________ Section Time: ______

Math 20C.
Final Exam
June 15, 2006

No calculators or any other devices are allowed on this exam.
Write your solutions clearly and legibly; no credit will be given for illegible solutions.
Read each question carefully. If any question is not clear, ask for clarification.
Answer each question completely, and show all your work.

1. (10 points) Find the plane through the point $P_0 = (2, 1, -1)$ which is perpendicular to the planes $2x + y - z = 3$ and $x + 2y + z = 2$.

$n_1 = \langle 2, 1, -1 \rangle$ and $n_2 = \langle 1, 2, 1 \rangle$: normal vectors should be parallel to the new plane $\Rightarrow$

$n_1 \times n_2$ is a normal vector.

$$n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = \left\langle \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}, -\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \right\rangle$$

$$= \langle 3, -3, 3 \rangle = 3\langle 1, -1, 1 \rangle$$

$\Rightarrow x - y + z = 2 - 1 - 1 = 0$

$\Rightarrow \boxed{x - y + z = 0}$

#	Score
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2. (8 points) Decide whether the $\lim_{(x,y)\to(0,0)} \frac{x^4 - y^2}{x^4 + y^2}$ exists. Give reasons your answer.

<u>Along the x-axis</u>

$y=0 \Rightarrow \lim_{x\to 0} \frac{x^4}{x^4} = 1.$

<u>Along the parabola $y=x^2$</u>

$\lim_{x\to 0} \frac{x^4 - x^4}{x^4 + x^4} = 0.$

Since $0 \neq 1$, $\lim_{(x,y)\to(0,0)} \frac{x^4 - y^2}{x^4 + y^2}$ does NOT exist.

3. (8 points) Does the function $f(x, y, z) = e^{3x+4y}\cos(5z)$ satisfy the Laplace equation $f_{xx} + f_{yy} + f_{zz} = 0$? Give reasons your answer.

$$f_x = 3e^{3x+4y}\cos(5z) = 3f \Rightarrow f_{xx} = 9f.$$

$$f_y = 4e^{3x+4y}\cos(5z) = 4f \Rightarrow f_{yy} = 16f.$$

$$f_z = -5e^{3x+4y}\sin(5z) \Rightarrow f_{zz} = -25e^{3x+4y}\cos(5z)$$

$$= -25f.$$

$$\Rightarrow f_{xx} + f_{yy} + f_{zz} = (9+16-25)f = 0.$$

4. (10 points) Find the linear approximation $L(x, y)$ of the function $f(x, y) = \sqrt{6 - x^2 - y^2}$ at the point $(1, 1)$. Use this approximation to estimate the value of $f(0.8, 1.1)$.

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$f_x = \frac{-x}{\sqrt{6-x^2-y^2}}$$

$$f_y = \frac{-y}{\sqrt{6-x^2-y^2}}$$

$$L(x,y) = 2 - \frac{1}{2}(x-1) - \frac{1}{2}(y-1)$$

$$= -\frac{x}{2} - \frac{y}{2} + 3 .$$

$$f(0.8, 1.1) \approx L(0.8, 1.1)$$

$$= -\frac{0.8}{2} - \frac{1.1}{2} + 3$$

$$= -0.4 - 0.55 + 3$$

$$= 2.05$$

5. (10 points) Find the local maxima, local minima and saddle points of the function $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$.

$$\nabla f = \langle 3x^2 + 6x \,,\, 3y^2 - 6y \rangle = \langle 0, 0 \rangle$$

$$\Rightarrow \begin{cases} 3x^2 + 6x = 0 \\ 3y^2 - 6y = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \ \underline{\text{or}}\ x = -2 \\ y = 0 \ \underline{\text{or}}\ y = 2 \end{cases}$$

Critical pts	$f_{xx} f_{yy} - (f_{xy})^2$	f_{xx}	result
$(0,0)$	$-$		$\Rightarrow$ Saddle pt
$(0,2)$	$+$	$+$	local min
$(-2,0)$	$+$	$-$	local max
$(-2,2)$	$-$		$\Rightarrow$ Saddle pt

$$\left.\begin{aligned} f_{xx} &= 6x + 6 \\ f_{yy} &= 6y - 6 \\ f_{xy} &= 0 \end{aligned}\right\} \Rightarrow D = 36\,(x+1)(y-1)$$

6. (10 points) Use Lagrange multipliers to find the maximum and minimum values of the function $f(x,y) = -\frac{1}{x} + \frac{1}{y}$ subject to the constraint $\frac{1}{x^2} + \frac{1}{y^2} = 1$.

First Solution.

$\frac{1}{x^2} + \frac{1}{y^2} = 1 \Rightarrow x \neq 0$ and $y \neq 0$

Let $u = \frac{1}{x}$ and $v = \frac{1}{y}$. Then we would like to find max and min of $-u+v$ subject to the constraints $u^2+v^2=1$ and $uv \neq 0$.

Since $u^2+v^2=1$ is bounded and closed, and $-u+v$ is continuous, it has a max and a min. We show neither occurs at $uv=0$, which implies that they are the desired values.

$$\left.\begin{array}{l} h(u,v) = -u+v \\ g(u,v) = u^2+v^2=1 \end{array}\right\} \Rightarrow \begin{cases} \nabla h = c\nabla g \\ g = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \langle -1,1\rangle = c\langle 2u,2v\rangle \\ u^2+v^2=1 \end{cases} \Rightarrow \begin{cases} c = \frac{-1}{2u} \quad (u\neq 0) \\ c = \frac{1}{2v} \quad (v \neq 0) \\ u^2+v^2=1 \end{cases}$$

$$\Rightarrow \begin{cases} u = -v \\ u^2+v^2=1 \end{cases} \Rightarrow 2u^2=1 \Rightarrow u = \pm\frac{\sqrt{2}}{2}$$

$$\Rightarrow (u,v) = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) \text{ or } \left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$$\Rightarrow \underbrace{f\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = -\sqrt{2}}_{\text{min}}, \quad \underbrace{f\left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \sqrt{2}}_{\text{max}}$$

Second Solution. If we assume that the max and the min exist, we can follow like this:

$$\begin{cases} \nabla f = c\,\nabla g \\ g = 1 \end{cases} \Rightarrow \begin{cases} \left\langle \frac{1}{x^2}, \frac{-1}{y^2} \right\rangle = c \left\langle \frac{-2}{x^3}, \frac{-2}{y^3} \right\rangle \\ \frac{1}{x^2} + \frac{1}{y^2} = 1 \end{cases}$$

$$\Rightarrow \begin{cases} c = \frac{-x}{2} = \frac{y}{2} \\ \frac{1}{x^2} + \frac{1}{y^2} = 1 \end{cases} \Rightarrow y = -x \Rightarrow \frac{2}{x^2} = 1$$

$$\Rightarrow x = \pm\sqrt{2} \Rightarrow (x,y) = (\sqrt{2}, -\sqrt{2}) \text{ or } (-\sqrt{2}, \sqrt{2})$$

$$\Rightarrow f(\sqrt{2}, -\sqrt{2}) = -\sqrt{2} \quad \text{min}$$

$$f(-\sqrt{2}, \sqrt{2}) = \sqrt{2} \quad \text{max}.$$

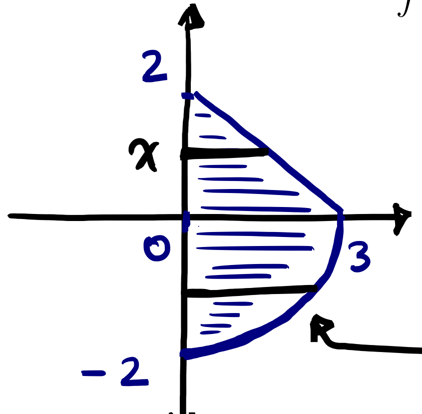
7. Consider the integral $\displaystyle\iint_D f(x,y)\,dA = \int_0^3 \int_{-2\sqrt{1-\frac{x^2}{3^2}}}^{2(1-\frac{x}{3})} f(x,y)\,dy\,dx.$

(a) (8 points) Sketch the region of integration.

(b) (8 points) Switch the order of integration in the above integral.

(c) (8 points) Compute the integral $\displaystyle\iint_D f(x,y)\,dA$ for the case $f(x,y) = xy$.

(a)



$$y = -2\sqrt{1-\left(\frac{x}{3}\right)^2}$$

$$\Rightarrow \left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

(b) $$\int_0^2 \int_0^{3(1-y/2)} f(x,y)\,dx\,dy + \int_{-2}^0 \int_0^{3\sqrt{1-(y/2)^2}} f(x,y)\,dx\,dy$$

$$y = 2\left(1-\frac{x}{3}\right)$$

$$\Rightarrow \frac{y}{2} = 1 - \frac{x}{3}$$

$$\Rightarrow x = 3\left(1-\frac{y}{2}\right)$$

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

$$\Rightarrow x = \pm 3\sqrt{1-\left(\frac{y}{2}\right)^2}$$

$x \geq 0$

$$\Rightarrow x = 3\sqrt{1-\left(\frac{y}{2}\right)^2}$$

(c) $$A(x) = \int_{-2\sqrt{1-\frac{x^2}{3^2}}}^{2(1-\frac{x}{3})} xy\,dy$$

$$= \frac{1}{2} x y^2 \Big|_{-2\sqrt{1-\frac{x^2}{3^2}}}^{2(1-x/3)}$$

$$= \frac{x}{2}\left[4\left(1-\frac{2}{3}x+\frac{x^2}{3^2}\right) - 4\left(1-\frac{x^2}{3^2}\right)\right]$$

$$= \frac{x}{2}\left(-\frac{8}{3}x + \frac{8}{9}x^2\right)$$

$$= -\frac{4}{3}x^2 + \frac{4}{9}x^3$$

$$\iint_D xy\, dA = \int_0^3 -\frac{4}{3}x^2 + \frac{4}{9}x^3\, dx$$

$$= \left(\frac{-4}{9}x^3 + \frac{1}{9}x^4\right)\Big|_0^3$$

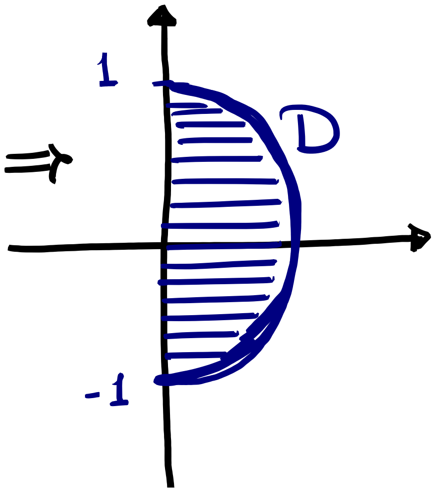
$$= -12 + 9 = -3 \,.$$

8. (10 points) Transform to polar coordinates and then evaluate the integral

$$I = \int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} \left(x^2 + y^2\right)^{3/2} dx\, dy.$$

$x = \sqrt{1-y^2}$

$\Rightarrow \left.\begin{array}{l} x^2+y^2=1 \\ x \geq 0 \end{array}\right\} \Rightarrow$



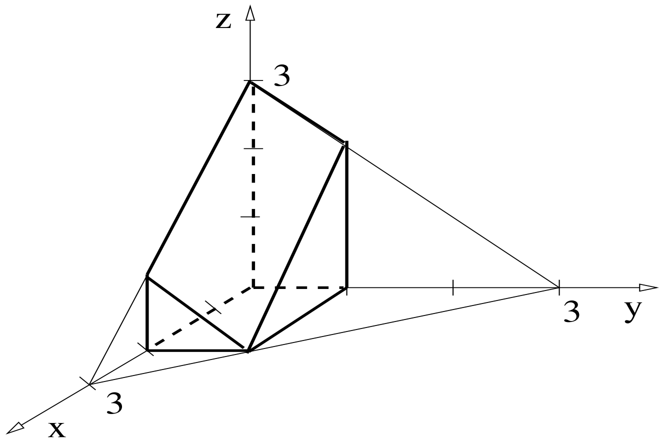
So $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and $0 \leq r \leq 1$.

$$\Rightarrow I = \int_{-\pi/2}^{\pi/2} \int_0^1 (r^2)^{3/2} \cdot r \, dr \, d\theta$$

$$A(\theta) = \int_0^1 r^4 \, dr = \frac{1}{5} r^5 \Big|_0^1 = \frac{1}{5}$$

$$\Rightarrow I = \int_{-\pi/2}^{\pi/2} \frac{1}{5} \, d\theta = \frac{\pi}{5}.$$

9. (10 points) Find the volume of a ~~parallelepiped~~ Solid whose base is a rectangle in the $z = 0$ plane given by $0 \leqslant y \leqslant 1$ and $0 \leqslant x \leqslant 2$, while the top side lies in the plane $x+y+z = 3$.



D : rectangle

$0 \leq x \leq 2$ and $0 \leq y \leq 1$

$$\Rightarrow \text{vol} = \iint_D f(x,y)\, dA$$

$$x+y+z=3 \Rightarrow z = 3-x-y$$

$$\Rightarrow f(x,y) = 3-x-y$$

$$\Rightarrow \text{vol} = \iint_D 3-x-y \; dA$$

$$= \int_0^2 \int_0^1 3-x-y \; dy\, dx$$

$$A(x) = \int_0^1 3-x-y \; dy = \left(3y - xy - \tfrac{1}{2}y^2\right)\Big|_0^1$$

$$= 3 - x - \tfrac{1}{2} = \tfrac{5}{2} - x$$

$$\Rightarrow \text{vol} = \int_0^2 \tfrac{5}{2} - x \; dx = \left(\tfrac{5}{2}x - \tfrac{1}{2}x^2\right)\Big|_0^2$$

$$= 5 - 2 = 3 .$$