Name: ______________________________ Student Number: ____________

Math 20C.
Midterm Exam 2
July 23, 2004

Read each question carefully, and answer each question completely.
Show all of your work. No credit will be given for unsupported answers.
Write your solutions clearly and legibly. No credit will be given for illegible solutions.

1. (8 points)

   Consider the function $f(x, y, z) = \sqrt{x + 2yz}$.

   (a) Find the gradient of $f(x, y, z)$.

$$\nabla f = \langle f_x, f_y, f_z \rangle = \left\langle \frac{1}{2\sqrt{x+2yz}}, \frac{2z}{2\sqrt{x+2yz}}, \frac{2y}{2\sqrt{x+2yz}} \right\rangle$$

$$= \left\langle \frac{1}{2\sqrt{x+2yz}}, \frac{z}{\sqrt{x+2yz}}, \frac{y}{\sqrt{x+2yz}} \right\rangle .$$

   (b) Find the directional derivative of f at $(0, 2, 1)$ in the direction given by $\langle 0, 3, 4\rangle$.

$$D_u f = \vec{u} \cdot \nabla f \quad \text{where } \vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 0,3,4\rangle}{\sqrt{9+16}} = \left\langle 0, \frac{3}{5}, \frac{4}{5} \right\rangle$$

(by part (a))

$$\nabla f(0,2,1) = \left\langle \frac{1}{4}, \frac{1}{2}, \frac{2}{2} \right\rangle = \left\langle \frac{1}{4}, \frac{1}{2}, 1 \right\rangle$$

$$D_u f = \left\langle 0, \frac{3}{5}, \frac{4}{5} \right\rangle \cdot \left\langle \frac{1}{4}, \frac{1}{2}, 1 \right\rangle = \frac{3}{10} + \frac{4}{5} = \frac{11}{10} .$$

   (c) Find the maximum rate of change of f at the point $(0, 2, 1)$.

The maximum rate of change of f at P_0 is $\|\nabla f(P_0)\|$. So the answer is $\|\nabla f(0,2,1)\| = \sqrt{\frac{1}{16} + \frac{1}{4} + 1}$

$$= \frac{\sqrt{1+4+16}}{4} = \frac{\sqrt{21}}{4} .$$

#	Score
1	
2	
3	
4	
Σ	

2. (8 points)

Find any value of the constant a such that the function $f(x,y) = e^{-ax}\cos(y) - e^{-y}\cos(x)$ is solution of Laplace's equation $f_{xx} + f_{yy} = 0$.

$$f_x = -a\, e^{-ax}\cos(y) + e^{-y}\sin(x)$$

$$f_{xx} = a^2\, e^{-ax}\cos(y) + e^{-y}\cos(x) \qquad \text{I}$$

$$f_y = -e^{-ax}\sin(y) + e^{-y}\cos(x)$$

$$f_{yy} = -e^{-ax}\cos(y) - e^{-y}\cos(x) \qquad \text{II}$$

I & II $\Rightarrow$

$$f_{xx} + f_{yy} = (a^2 - 1)\, e^{-ax}\cos(y)$$

So $f_{xx} + f_{yy} = 0$ if and only if

$$a^2 - 1 = 0 \Rightarrow a = \pm 1.$$

3. (8 points)

Let $f(x,y) = 12xy - 2x^3 - 3y^2$.

(a) Find all the critical (stationary) points of f.

(b) For each critical point of f, determine whether f has a local maximum, local minimum, or saddle point at that point.

a) $\nabla f = \langle f_x, f_y \rangle = \langle 12y - 6x^2, 12x - 6y \rangle = \langle 0,0 \rangle$

$$\Rightarrow \begin{cases} 12y - 6x^2 = 0 \\ 12x - 6y = 0 \end{cases} \Rightarrow \begin{cases} 2y = x^2 \\ 2x = y \end{cases}$$

$\Rightarrow 4x = x^2 \Rightarrow x = 0$ or $x = 4$

$\Rightarrow$ critical pts are $(0,0)$ and $(4,8)$.

(b) $$\left.\begin{array}{l} f_{xx} = -12x \\ f_{xy} = 12 \\ f_{yy} = -6 \end{array}\right\} \Rightarrow D = f_{xx} f_{yy} - f_{xy}^2 = 72x - 144 = 72(x-2).$$

Critical pts	D	f_{xx}	Result
$(0,0)$	$-$	0	Saddle pt
$(4,8)$	$+$	$-$	local max

NOT PART OF OUR EXAM

4. (8 points)

Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = x^2 + y^2$ subject to the constraint $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$.

Let $g(x,y) = \frac{1}{4}x^2 + \frac{1}{9}y^2$. At the max and the min we have that $\nabla f = c\nabla g$ for some c.

$\nabla f = \langle 2x, 2y \rangle$

$\nabla g = \langle \frac{x}{2}, \frac{2y}{9} \rangle$

$\Rightarrow \langle 2x, 2y \rangle = c\langle \frac{x}{2}, \frac{2y}{9} \rangle$

$$\Rightarrow \begin{cases} 2x = c\frac{x}{2} & \text{I} \\ 2y = c\frac{2y}{9} & \text{II} \end{cases}$$

I $\Rightarrow$ $x = 0$ or $c = 4$.

If $x = 0$, then $y = \pm 3$ since $\frac{x^2}{4} + \frac{y^2}{9} = 1$

If $c = 4$, then, by II, $y = 0$. And so $x = \pm 2$ as $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

$$\left.\begin{array}{l} f(0, \pm 3) = 9 \\ f(\pm 2, 0) = 4 \end{array}\right\} \Rightarrow \text{the max is 9 and the min is 4.}$$