Name: ____________________ PID: ____________

TA: ________________ Sec. No: ____ Sec. Time: ____

Math 20C.
Midterm Exam 1
April 25, 2007

Turn off and put away your cell phone.
You may use any type of handheld calculator; no other devices are allowed on this exam.
You may use one page of notes, but no books or other assistance on this exam.
Read each question carefully, answer each question completely, and show all of your work.
Write your solutions clearly and legibly; no credit will be given for illegible solutions.
If any question is not clear, ask for clarification.

1. (4 points) Let $\mathbf{a} = \langle -2, 1, 8\rangle$ and let $\mathbf{u} = \langle s, s^2, -1\rangle$. Find the values of s for which $\mathbf{a}$ and $\mathbf{u}$ are orthogonal.

$\vec{a} \perp \vec{u}$ if and only if $\vec{a}\cdot\vec{u} = 0$

$\vec{a}\cdot\vec{u} = (-2)(s) + (1)(s^2) + (8)(-1)$

$= -2s + s^2 - 8 = 0$

So $s^2 - 2s + 1 = 9$.

$\Rightarrow (s-1)^2 = 9$

$\Rightarrow s - 1 = \pm 3$

$\Rightarrow s = 4$ or -2.

#	Points	Score
1	4	
2	4	
3	6	
4	10	
5	6	
Σ	30	

2. Let $\mathbf{a} = \langle 1, 1, 1 \rangle$ and $\mathbf{b} = \langle -2, -2, 1 \rangle$.

(a) (2 points) Find two vectors that are orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Any vector parallel to $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ -2 & -2 & 1 \end{vmatrix} = \left\langle \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ -2 & -2 \end{vmatrix} \right\rangle$$

$$= \langle (1)(1) - (1)(-2), -(1)(1) + (1)(-2), (1)(-2) - (1)(-2) \rangle$$

$$= \langle 3, -3, 0 \rangle.$$

So $\langle 3, -3, 0 \rangle$ and $\langle -3, 3, 0 \rangle$ are orthogonal to $\vec{a}, \vec{b}$.

(b) (2 points) Find the sine of the angle between $\mathbf{a}$ and $\mathbf{b}$.

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \, \|\vec{b}\| \sin\theta$$

$$\|\vec{a} \times \vec{b}\| = \sqrt{9 + 9 + 0} = 3\sqrt{2}.$$

$$\|\vec{a}\| = \sqrt{1 + 1 + 1} = \sqrt{3}.$$

$$\|\vec{b}\| = \sqrt{4 + 4 + 1} = 3.$$

So $\sin\theta = \dfrac{3\sqrt{2}}{3\sqrt{3}} = \dfrac{\sqrt{2}}{\sqrt{3}} = \dfrac{\sqrt{6}}{3}.$

3. (6 points) Find an equation for the plane containing the lines $\boldsymbol{\lambda}(t) = \langle 1, 3, 5\rangle + t\langle 1, 4, 7\rangle$ and $\boldsymbol{\mu}(t) = \langle 1, 3, 5\rangle + t\langle 2, -1, 4\rangle$.

Since it contains λ and μ, it is parallel to the vectors $\langle 2,-1,4\rangle$ and $\langle 1,4,7\rangle$. Hence

$$\vec{n} = \langle 2,-1,4\rangle \times \langle 1,4,7\rangle$$

is a normal of the considered plane.

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 4 \\ 1 & 4 & 7 \end{vmatrix} = \left\langle \begin{vmatrix} -1 & 4 \\ 4 & 7 \end{vmatrix}, -\begin{vmatrix} 2 & 4 \\ 1 & 7 \end{vmatrix}, \begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} \right\rangle$$

$$= \langle -7-16, -14+4, 8+1\rangle$$

$$= \langle -23, -10, 9\rangle$$

The considered plane also passes through $(1,3,5)$.

Hence its equation is

$$-23x - 10y + 9z = (-23)(1) - (10)(3) + (9)(5)$$

$$= -23 - 30 + 45$$

$$= -8$$

So $-23x - 10y + 9z = -8$.

4. A particle's position function is $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), 2t \rangle$ for $0 \le t \le 2\pi$.

(a) (2 points) Find the particle's velocity $\mathbf{v}(t)$ and speed $|\mathbf{v}(t)|$ as a function of time t.

$$v(t) = r'(t) = \langle -2\sin t, 2\cos t, 2 \rangle$$

$$\|v(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t + 4} = \sqrt{8} = 2\sqrt{2}.$$ [will not be in our exam 1.]

(b) (2 points) Find the particle's acceleration $\mathbf{a}(t)$ as a function of time t.

$$a(t) = v'(t) = \langle -2\cos t, -2\sin t, 0 \rangle$$

(c) (3 points) Find the angle between the particle's position $\mathbf{r}(t)$ and acceleration $\mathbf{a}(t)$ as a function of time t.

$$a(t) \cdot r(t) = \|a(t)\| \, \|r(t)\| \cos\theta(t)$$

$$\Rightarrow \langle -2\cos t, -2\sin t, 0 \rangle \cdot \langle 2\cos t, 2\sin t, 2t \rangle =$$

$$-4\cos^2 t - 4\sin^2 t = -4.$$

$$\|r(t)\| = \sqrt{4\cos^2 t + 4\sin^2 t + 4t^2} = \sqrt{4+4t^2} = 2\sqrt{1+t^2}$$

$$\|a(t)\| = \sqrt{4\cos^2 t + 4\sin^2 t} = 2$$ (Contine at the bottom)

(d) (3 points) Determine how far the particle traveled during the time interval $0 \le t \le 2\pi$.

$$\text{Total distance} = \int_0^{2\pi} \|r'(t)\| \, dt$$ [will not be in our exam 1.]

$$= \int_0^{2\pi} 2\sqrt{2} \, dt$$

$$= 4\sqrt{2}\,\pi.$$

$$\cos\theta(t) = \frac{-4}{(2\sqrt{1+t^2})(2)} = \frac{-1}{\sqrt{1+t^2}}.$$

$$\Rightarrow \theta(t) = \operatorname{Arccos}\left(\frac{-1}{\sqrt{1+t^2}}\right).$$

[will not be in our exam 1.]

5. (a) (3 points) Evaluate $\lim_{(x,y)\to(0,0)} \frac{2(x^2+y^2)}{\sqrt{x^2+y^2+1}-1}$.

Let's use polar coordinates: $x = r\cos\theta$ and $y = r\sin\theta$.

So

$$\lim_{(x,y)\to(0,0)} \frac{2(x^2+y^2)}{\sqrt{x^2+y^2+1}-1} = \lim_{r\to 0^+} \frac{2r^2}{\sqrt{r^2+1}-1}$$

(Since the function is independent of θ.)

$$\lim_{r\to 0^+} \frac{2r^2(\sqrt{r^2+1}+1)}{[(r^2+1)-1]} = \lim_{r\to 0^+} 2(\sqrt{r^2+1}+1) = 4.$$

(b) (3 points) Explain why $\lim_{(x,y)\to(0,0)} \frac{x^2}{2x^2+y^2}$ does not exist.

If it exists, we should get the same limit as we go to $(0,0)$ along any line (which passes the origin.) For instance $y = x$ and $y = 2x$.

$$\lim_{x\to 0} \frac{x^2}{2x^2+x^2} = \lim_{x\to 0} \frac{1}{3} = \frac{1}{3} \quad \text{(along the line } y=x.)$$

$$\lim_{x\to 0} \frac{x^2}{2x^2+4x^2} = \lim_{x\to 0} \frac{1}{6} = \frac{1}{6} \quad \text{(along the line } y=2x.)$$

Since $\frac{1}{3} \neq \frac{1}{6}$, $\lim_{(x,y)\to(0,0)} \frac{x^2}{2x^2+y^2}$ does not exist.