Lecture 28: Order of integration

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Sometimes we have to choose the order of integration carfully:

<u>Ex.</u> Find $\int_0^1 \int_{x^{2/3}}^1 x e^{y^4}\, dy\, dx$.

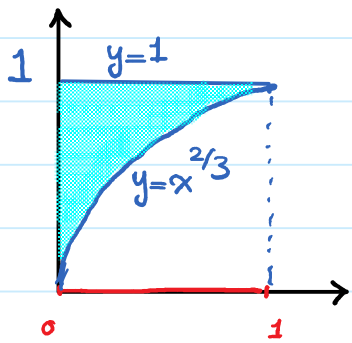
<u>Solution</u>. If we try to compute this directly, we have to compute $\int_{x^{2/3}}^1 x e^{y^4}\, dy$ first. But $\int x e^{y^4} dy$ is not a "nice" function. In such cases, we should switch the order of integration. Sometimes the order is (either easier or) doable. To switch the order of integration one has to

① Sketch the region ② setup in the new order.

We start by sketching:

$$\int_0^1 \int_{x^{2/3}}^1 \underbrace{\cdots}\, dy\, dx$$

for this part we do not care what the function is.



It means $0 \le x \le 1$ and, for a given x, $x^{2/3} \le y \le 1$

To setup the integral in the new order $\int \int \cdots\, dx\, dy$

First we have to project to the y-axis

Then, for a given y, find the range of x.

Lecture 28: Order of integration; volume of solid

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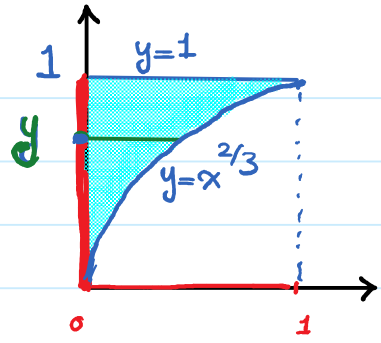

As we can see $0 \le y \le 1$. Now for a fixed y, x is varying from the y-axis to the curve $y = x^{2/3}$.

y-axis is $x = 0$ and $y = x^{2/3} \Rightarrow x = y^{3/2}$. So, for a fixed y, $0 \le x \le y^{3/2}$. Overall we get

$$\int_0^1 \int_{x^{2/3}}^1 x e^{y^4}\, dy\, dx = \iint_D x e^{y^4}\, dA = \int_0^1 \int_0^{y^{3/2}} x e^{y^4}\, dx\, dy$$

$$= \int_0^1 \left(\frac{x^2}{2} e^{y^4}\right)\Big|_{x=0}^{x=y^{3/2}} dy$$

$$= \int_0^1 \frac{(y^{3/2})^2}{2} e^{y^4}\, dy = \int_0^1 \frac{1}{2} y^3 e^{y^4}\, dy$$

$$= \int_0^1 \left(\frac{1}{2}\right) e^u \frac{du}{4}$$

substitution rule
$u = y^4 \Rightarrow du = 4y^3\, dy$

$$= \frac{1}{8} e^u \Big|_0^1 = \frac{1}{8}(e-1).$$

Ex. Compute the volume of the solid enclosed by

$$y = x^2 - 1, \quad y = 1 - x^2 \quad \text{and} \quad 0 \le z \le x^2.$$

Solution. Recall volume of the solid where (x,y) is in a region D and $0 \le z \le f(x,y)$ is equal to $\iint_D f(x,y)\, dA$.

Lecture 28: Volume of a solid

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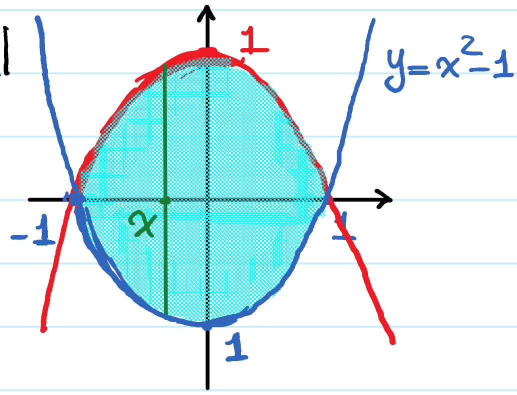
So volume of the solid enclosed by $y=x^2-1$, $y=1-x^2$, and $0\le z\le x^2$ is equal to $\iint_D x^2\, dA$ where D is the region enclosed by $y=x^2-1$ and $y=1-x^2$.

To write $\iint_D \dots dA$ as iterated integrals we have to sketch D. So we have to sketch $y=x^2-1$ and $y=1-x^2$.

This region has an easier vertical description: $-1\le x\le 1$

$$x^2-1\le y\le 1-x^2$$



So: volume $= \displaystyle\int_{-1}^{1}\int_{x^2-1}^{1-x^2} x^2\, dy\, dx$

$$= \int_{-1}^{1} (x^2 y)\Big|_{y=x^2-1}^{y=1-x^2} dx$$

$$= \int_{-1}^{1} x^2\left[(1-x^2)-(x^2-1)\right] dx = 2\int_{-1}^{1} x^2-x^4\, dx$$

$$= 2\left(\frac{x^3}{3}-\frac{x^5}{5}\right)\Big|_{-1}^{1} = 2\left[\left(\frac{1}{3}-\frac{1}{5}\right)-\left(\frac{-1}{3}-\frac{-1}{5}\right)\right]$$

$$= (4)\left(\frac{2}{15}\right) = \frac{8}{15}.$$

Lecture 28: Recall polar coordinates

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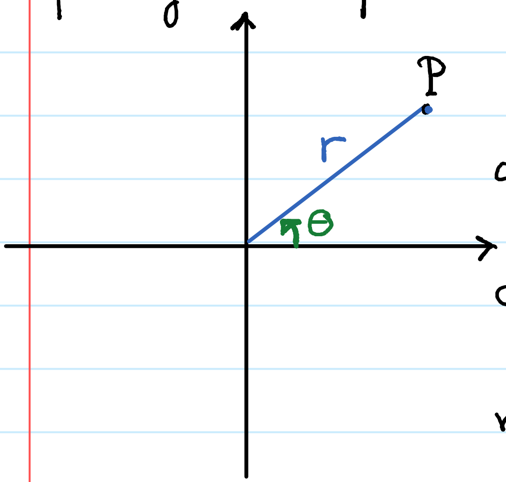
In single-variable integrations you learned a lot of techniques to deal with complicated functions. A double integral has two sources of complications: ① function ② region.

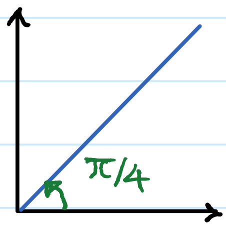
For some of the regions one has to use polar coordinates. Let's quickly recall polar coordinates:



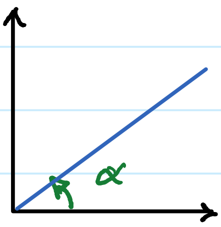
(r, θ) is the polar coordinate of P where r is the distance from P to the origin. And θ is the angle that OP makes with the positive direction of the x-axis.

<u>Ex.</u> What is $\theta = \frac{\pi}{4}$?

<u>Solution</u>. It is the following half-line.



In general, for a given α, $\theta = \alpha$ is a half-line



($\theta = 0$: positive direction of x-axis;
$\theta = \pi/2$: positive direction of y-axis; $\theta = \pi$: negative direction of x-axis.)

Lecture 28: Recall polar coordinates

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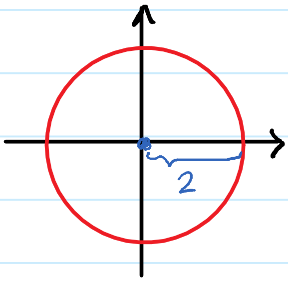
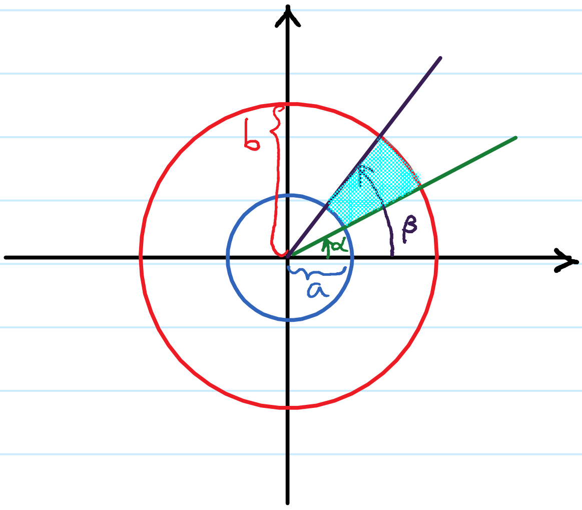
Ex. $r=2$ is an equation of a circle centered at the origin with radius 2.



The region $a \le r \le b$, $\alpha \le \theta \le \beta$ is the following:



We will see how polar coordinates can help us in computing some double integrals.