Lecture 27: Double integral over a more general region

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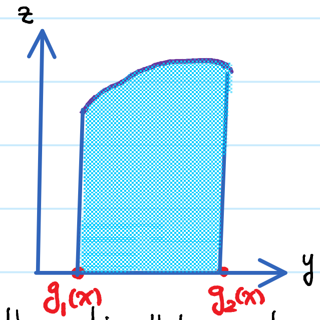
So far we learned how to integrate over a rectangle. Using the same idea of first fixing $x=x_0$ and computing the area $A(x_0)$ of the section of the solid above D and under the graph of $f(x,y)$ we get the following:

$$\iint_D f(x,y)\,dA = \int_{\substack{\text{range} \\ \text{for } x}} A(x)\,dx$$

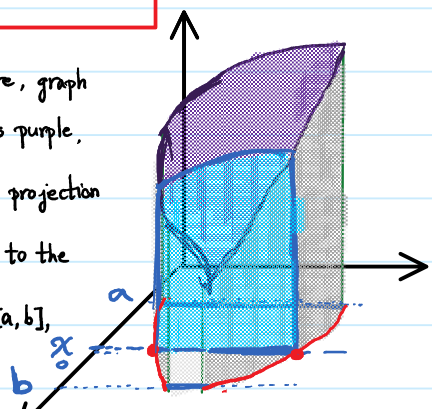
One can find this range by projection D to the x-axis. (looking at its shadow.)

Section of the solid in $x=x_0$



As you can see the section that we get is above parts of line $x=x_0$, $z=0$ which is between the red curves. And it is under the purple curve which is $z=f(x_0,y)$. So if the red curves are graphs $y=g_1(x)$ and $y=g_2(x)$, then we get the above figure.

In this figure, graph of $f(x,y)$ is purple, The range of projection of the solid to the x-axis is $[a,b]$, The region D is enclosed by the red curves and $a\le x\le b$,



And for any point in the solid we have $0\le z\le f(x,y)$, which means points above D and under the graph.

As you can see in the above figures, if D is discribed by $a\le x\le b$, $g_1(x)\le y\le g_2(x)$, then, for any x, $A(x)$ is

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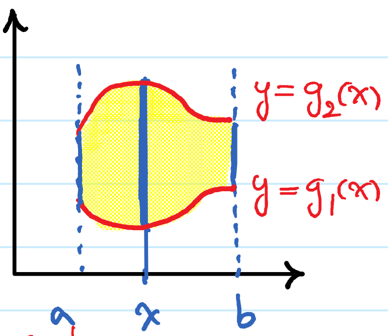
$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy .$$ Therefore we get

$$\boxed{\begin{array}{l} \text{If a region } D \text{ in the } xy\text{-plane is given by} \\ a \le x \le b,\ g_1(x) \le y \le g_2(x), \text{ then} \\ \displaystyle\iint_D f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx . \end{array}}$$

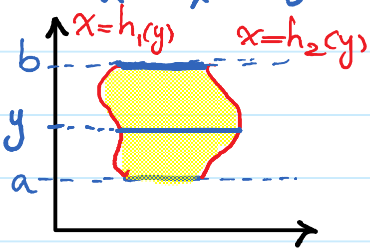
By a similar argument (this time first fixing y), we get:

$$\boxed{\begin{array}{l} \text{If a region } D \text{ in the } xy\text{-plane is given by} \\ a \le y \le b,\ h_1(y) \le x \le h_2(y) \text{ then} \\ \displaystyle\iint_D f(x,y)\, dA = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy . \end{array}}$$

- For the top box region D looks like (vertical region or y-simple)



- For the bottom box region D looks like (horizontal region or x-simple)



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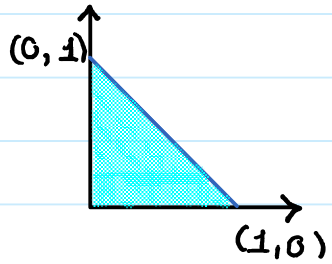
Ex. Find $\iint_D xy \, dA$ where D is the following triangle.



Solution. We can think about this region as a vertical or horizontal region.

Let's view D as a vertical region. To setup the iterated integrals, you have to do the following:

To find ranges for vertical regions (y-simple)

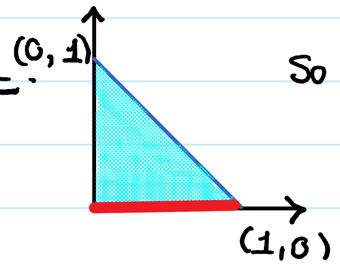
Step 1. Find the projection of the given region to the x-axis, in order to find the range for x.

You should get two fixed numbers.

Step 2. For a given x, sketch a line parallel to the y-axis, in order to find the range for y.

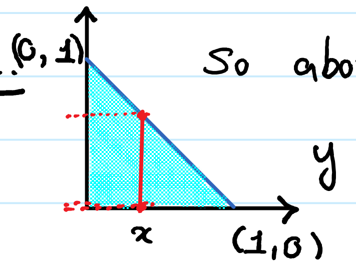
You should get functions in terms of x. (The bounds might be contsants)

Step 1.



So $0 \leq x \leq 1$.

Step 2.



So above x y ranges from 0 till we reach to the

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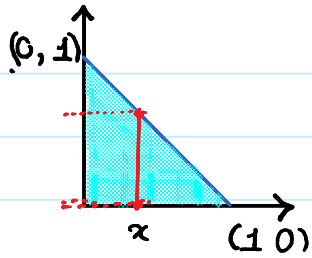
which connects $(1,0)$ to $(0,1)$.



Equation of this line is $x+y=1$.

So $0 \le y \le 1-x$.

Hence
$$\iint_D xy \, dA = \int_0^1 \int_0^{1-x} xy \, dy \, dx$$

$$= \int_0^1 \left. \frac{xy^2}{2} \right|_{y=0}^{y=1-x} dx$$

$$= \int_0^1 \frac{x}{2}(1-x)^2 \, dx = \frac{1}{2}\int_0^1 x - 2x^2 + x^3 \, dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2} - 2\frac{x^3}{3} + \frac{x^4}{4} \right)\Bigg|_0^1$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) = \frac{6-8+3}{24}$$

$$= \frac{1}{24}.$$

Warning • If you set up iterated integrals with order dy dx

then $\int \int \cdots \, dy \, dx$

Here should be constants

Here cannot depend on y

• In general $\int \cdots \, dx$ → cannot depend on x

$\int \cdots \, dy$ → cannot depend on y

Lecture 27: Switching the order of integration

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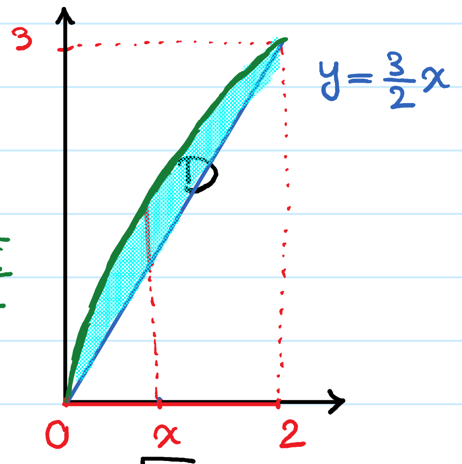
Ex. D is described as $0 \le x \le 2$ and $\frac{3}{2}x \le y \le 3\sqrt{\frac{x}{2}}$.

(a) Sketch D.

(b) Compute $\iint_D dA$ by viewing D as a vertical region.

(c) Switch the order of integration. (Only setup the iterated integrals.)

Solution. (a)



(b) $$\iint_D dA = \int_0^2 \int_{\frac{3}{2}x}^{3\sqrt{\frac{x}{2}}} dy\, dx = \int_0^2 y \Big|_{\frac{3}{2}x}^{3\sqrt{\frac{x}{2}}} dx$$

$$= \int_0^2 \left(3\sqrt{\frac{x}{2}} - \frac{3}{2}x\right) dx = \left(\frac{3}{\sqrt{2}} \cdot \frac{2}{3} x^{\frac{3}{2}} - \frac{3}{2} \cdot \frac{1}{2} x^2\right)\Big|_0^2$$

$$= \sqrt{2} \cdot 2^{3/2} - \frac{3}{4} \cdot 2^2 = 2^{\frac{1}{2}+\frac{3}{2}} - 3 = 4 - 3 = 1.$$

(c) We want to write the above integral as

$$\int \int \quad dx\, dy$$

We use the steps mentioned earlier: project to y-axis

Lecture 27: Switch the order of integration

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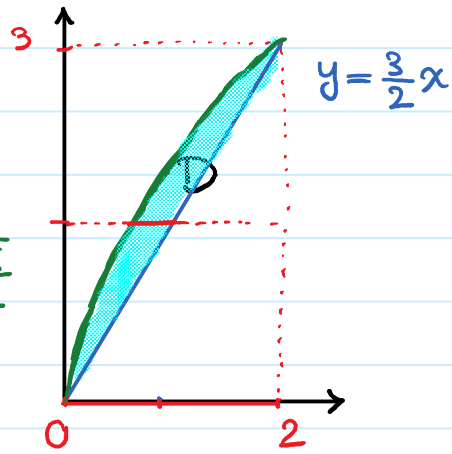
and we get $0 \le y \le 3$.



Now, for a fixed y, we find the range for x.

For a given y, x ranges from the parabola $y=3\sqrt{\frac{x}{2}}$ to the line $y=\frac{3}{2}x$. Since we are looking for (a range of) x, we should solve for x.

We get $y=3\sqrt{\frac{x}{2}} \Rightarrow \frac{y}{3}=\sqrt{\frac{x}{2}} \Rightarrow \frac{y^2}{9}=\frac{x}{2}$

$$\Rightarrow x=\frac{2}{9}y^2$$

and $y=\frac{3}{2}x \Rightarrow x=\frac{2}{3}y$.

Hence $\frac{2}{9}y^2 \le x \le \frac{2}{3}y$ (for a given y).

So $\iint_D dA = \int_0^3 \int_{\frac{2}{9}y^2}^{\frac{2}{3}y} dx\, dy$.