Lecture 26: Double integral

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After differentiation, we would like to study integration of multivariable functions.

In the single-variable case, $\int_a^b f(x)\,dx$, when $f(x) \geq 0$ and $a \leq b$, gives us the area of the region

- above the interval $[a,b]$ in the x-axis,
- under the graph of $f(x)$.

For two-variable functions, we use a similar idea to define the double integral of f over a region D:

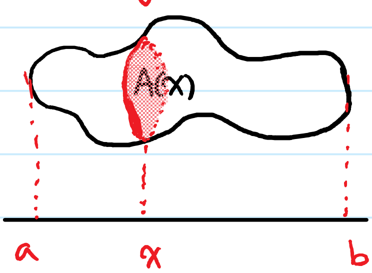
For a region D in the xy-plane, let

$$\iint_D f(x,y)\,dA = \text{volume of the solid above } D \text{ and under the graph of } f(x,y).$$

when $f(x,y) \geq 0$.

How can we compute a double integral?

For a given solid, its volume can be determined using the area of slices: volume $= \int_a^b A(x)\,dx$



Lecture 26: Cavalieri's principle and iterated integrals

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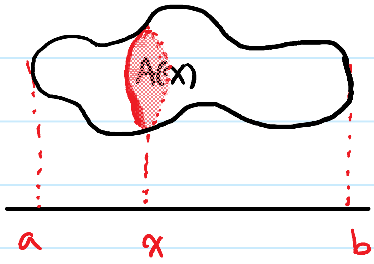
This idea goes back to Cavalieri. Let us repeat the formula here

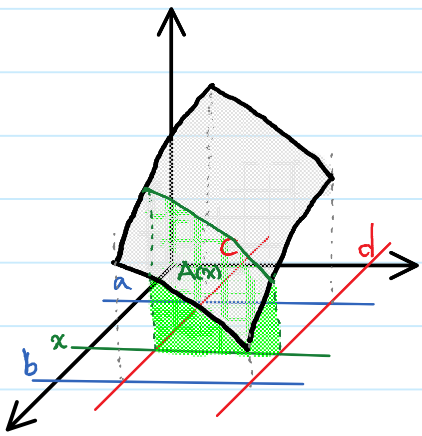
$$\text{volume} = \int \text{area of sections}.$$

$$= \int_a^b A(x)\, dx.$$



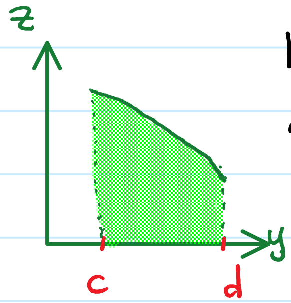
Using this formula, let's try to compute $\iint_D f(x,y)\, dA$ when D is the rectangle $[a,b] \times [c,d]$.

$\iint_D f(x,y)\, dA$ = volume of the solid above D under graph of $f(x,y)$.



$$= \int_a^b A(x)\, dx.$$

$$\boxed{A(x_0) = \int_c^d f(x_0, y)\, dy}$$



In the plane $x = x_0$, slice is the region above $[c,d]$ and under graph of $f(x_0, y)$

So we get $\displaystyle\iint_D f(x,y)\, dA = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx.$

Lecture 26: Double and iterated integrals

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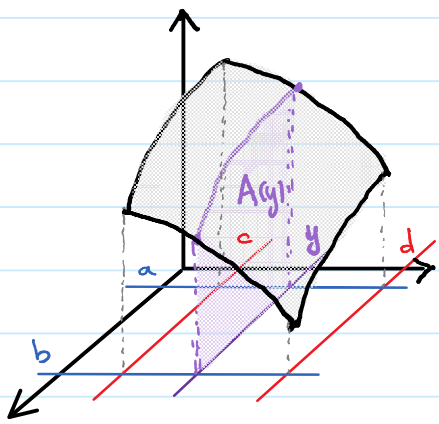
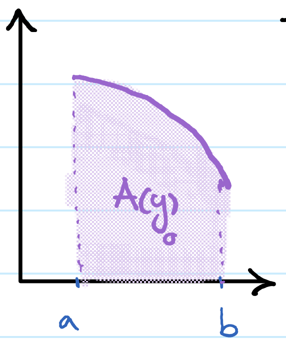
By slicing this solid using planes parallel to the xz-plane we get:

$$\iint_D f(x,y)\,dA = \int_c^d A(y)\,dy$$



$$\boxed{A(y_0) = \int_a^b f(x,y_0)\,dx}$$

So $\displaystyle\iint_D f(x,y)\,dA = \int_c^d \left(\int_a^b f(x,y)\,dx\right)dy$.



The slice by plane $y=y_0$ is the region above $[a,b]$ and under $f(x,y_0)$

These are parts of Fubini's theorem.

Overall we get that a double integral (over a rectangle) can be computed using iterated integrals:

$$\iint_D f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

where D is the rectangle $a \le x \le b$ and $c \le y \le d$.

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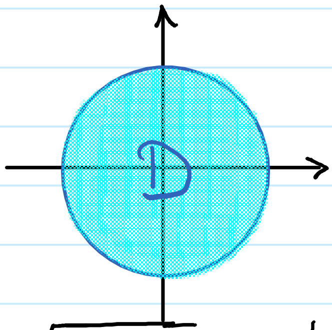
Ex. Compute $\iint_D \sqrt{1-x^2-y^2}\, dA$ where D is the unit disk centered at the origin



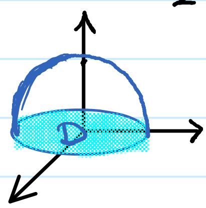
Solution. $\iint_D \sqrt{1-x^2-y^2}\, dA$

is the volume of the solid under $\sqrt{1-x^2-y^2}$ and above the region D. So next we visualize $z=\sqrt{1-x^2-y^2}$.

First notice $z \geq 0$ as it is square-root of a number.

Second $z^2 = 1-x^2-y^2$. So $x^2+y^2+z^2=1$ which is a sphere centered at the origin with radius 1. Hence overall we have $z=\sqrt{1-x^2-y^2}$ is the upper-half of the sphere of radius 1 centered at the origin (which is above D). So we have to find volume of half of a ball of radius 1:

$$\iint_D \sqrt{1-x^2-y^2}\, dA = \frac{1}{2} \text{volume of a ball of radius } 1 = \frac{1}{2}\left(\frac{4}{3}\pi\right) = \frac{2}{3}\pi .$$



Lecture 26: Double integrals; examples

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Ex. Compute $\iint_D xy\, dA$ where D is the rectangle $1 \le x \le 2$ and $3 \le y \le 4$.

Solution. $\iint_D xy\, dA = \int_1^2 \underbrace{\int_3^4 xy\, dy}_{\text{first should be computed:}}\, dx$

$$\int_3^4 xy\, dy = x\, y^2/2 \Big|_{y=3}^{y=4} = \frac{x}{2}(4^2 - 3^2) = \frac{7x}{2}.$$

we treat x as a constant

Hence $\iint_D xy\, dA = \int_1^2 \frac{7x}{2}\, dx = \frac{7x^2}{4}\Big|_1^2 = \frac{7}{4}(4-1)$

$$= \frac{21}{4}.$$

Ex. Compute $\int_0^1 \int_0^1 \frac{y}{1+xy}\, dy\, dx$.

Solution To solve this problem directly, one has to compute $\int \frac{y}{1+xy}\, dy$. One can use the substitution $v = 1+xy$, and so $dv = x\, dy$ and $\frac{y}{1+xy}\, dy = \frac{y}{v} \cdot \frac{dv}{x}$

Since $v = 1+xy$, we have $y = \frac{v-1}{x}$. Therefore

$\frac{y}{1+xy}\, dy = \frac{v-1}{v} \cdot \frac{dv}{x^2} = \frac{1}{x^2}\left(1 - \frac{1}{v}\right) dv$. Now you can

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compute $\int \frac{1}{x^2}\left(1-\frac{1}{v}\right) dv$ and then continue with the 2^{nd} integral. As you can see, this method is a bit involve.

On the other hand, by Fubini's theorem

$$\int_0^1 \int_0^1 \frac{y}{1+xy}\, dy\, dx = \iint_D \frac{y}{1+xy}\, dA = \int_0^1 \int_0^1 \frac{y}{1+xy}\, dx\, dy$$

So we can start with

$$\int_0^1 \frac{y}{1+xy}\, dx = \int_{u(0)}^{u(1)} \frac{du}{u} = \int_1^{1+y} \frac{du}{u} = \ln|1+y|$$

$u = 1+xy$ think about as a function of $x \Rightarrow du = y\, dx$

Next we have to compute

$t = 1+y$
$\Rightarrow dt = dy$

$$\int_0^1 \ln|1+y|\, dy = \int_1^2 \ln t\, dt = (t\ln t - t)\Big|_1^2 = (2\ln 2 - 2) - (-1) = 2\ln 2 - 1.$$

(Recall. Integration-by-part $\int u\, dv = uv - \int v\, du$ can be used to compute $\int \ln t\, dt$. Let $u = \ln t$ and $dv = dt$. So $du = \frac{dt}{t}$ and $v = t$. Hence

$$\int \ln t\, dt = uv - \int v\, du = t\ln t - \int t \cdot \frac{dt}{t} = t\ln t - t + C)$$

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So in any double integral or iterated integral one should check both orders of integrations to find out which one is easier from computational point of view.