Lecture 25: Antiderivative of a vector valued function

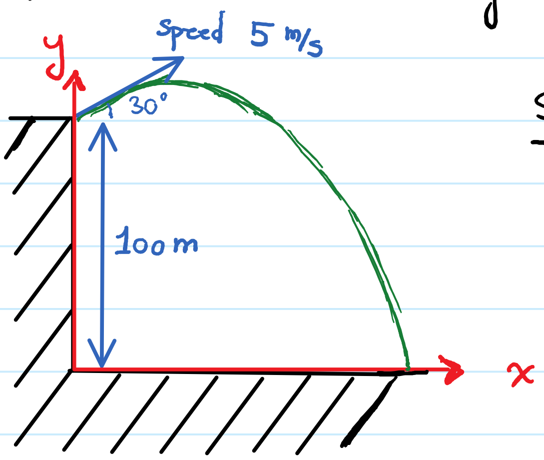
Monday, November 21, 2016 8:19 AM

We have seen how derivative of a vector-valued function helps us compute velocity and acceleration of a moving particle.

Ex. A stone is thrown at a speed 5 m/s with angle 30° (compared to the horizontal line) from the top of a cliff 100 m high.

(a) Find the position of the stone after t seconds.

(b) When does it hit the ground?



Solution. Initial point $= (0, 100)$

$= \vec{r}(0)$

• Initial velocity $=$ (speed) (directional vector)

$= 5\,(\cos 30^\circ, \sin 30^\circ)$

$= 5\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \left(\frac{5\sqrt{3}}{2}, \frac{5}{2}\right)$

$= \vec{r}\,'(0) = \vec{v}(0)$

• acceleration $=$ from gravitational force $= (0, -g) = \vec{r}\,''(t) = \vec{v}\,'(t)$

(g is the earth gravitational constant ≈ 10 m/s^2).

So $\vec{v}\,'(t) = (0, -g)$. Therefore taking anti-derivative of both sides we get: $\vec{v}(t) - \vec{v}(0) = \left(\int_0^t 0\, du, \int_0^t -g\, du\right) = (0, -gt)$.

Thus $\vec{v}(t) = \vec{v}(0) + (0, -gt) = \left(\frac{5\sqrt{3}}{2}, \frac{5}{2} - gt\right) = \vec{r}\,'(t)$.

Again taking anti-derivative of both sides we get:

$$\vec{r}(t) - \vec{r}(0) = \left(\int_0^t \frac{5\sqrt{3}}{2}\,du, \int_0^t \frac{5}{2} - gt\right)$$

$$= \left(\frac{5\sqrt{3}}{2}t, \frac{5}{2}t - \frac{gt^2}{2}\right).$$

Hence $\vec{r}(t) = \left(\frac{5\sqrt{3}}{2}t, 100 + \frac{5}{2}t - \frac{gt^2}{2}\right)$.

(b) To find when it hits the ground we should find out when the y-component is 0.

(Let's approximate $g \approx 10$)

$$-5t^2 + \frac{5}{2}t + 100 = 0 \Rightarrow t^2 - \frac{1}{2}t - 20 = 0.$$

$$\Rightarrow \left(t - \frac{1}{4}\right)^2 = 20 + \frac{1}{16} = \frac{321}{16} \Rightarrow t - \frac{1}{4} = \pm\sqrt{\frac{321}{16}}.$$

$$\Rightarrow t = \frac{1 \pm \sqrt{321}}{4}.$$

Only the positive value is acceptable: $t = \frac{1+\sqrt{321}}{4} \approx 4.5$ s.

We used the following twice in the above example:

$$\int_a^b (x(t), y(t), z(t))\,dt = \left(\int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt\right)$$

Lecture 25: Total length traveled by a particle

Monday, November 21, 2016 8:59 AM

• Suppose a moving particle at time t is at $\vec{r}(t)$. We'd like to find the total distance traveled by this particle in $a \le t \le b$.

It is the same as asking what the length of the curve parametrized by $\vec{r}(t)$ is, for $a \le t \le b$.

If the particle is moving by a constant speed s_0, then the answer is $s_0(b-a)$. In general, we get

$$\boxed{\begin{array}{l}\text{Total distance}\\ \text{traveled } a\le t\le b\end{array} = \int_a^b \text{speed}(t)\, dt = \int_a^b \|\vec{r}\,'(t)\|\, dt}$$

Similarly

$$\boxed{\begin{array}{l}\text{length (it is often called arc-length)}\\ \text{of the curve parametrized by}\\ \vec{r}(t), \text{ for } a\le t\le b\end{array} = \int_a^b \|\vec{r}\,'(t)\|\, dt}$$

Lecture 25: Arc-length

Monday, November 21, 2016 9:13 AM

Ex. Find the arc-length of $\vec{r}(t) = (\sin(3t), \cos(3t), 2t^{3/2})$

for $0 \le t \le u$.

Solution. We have

$$\text{arc-length of } \vec{r}(t) = \ell(u) = \int_0^u \|\vec{r}\,'(t)\|\, dt.$$

$$\vec{r}\,'(t) = (3\cos(3t), -3\sin(3t), 3t^{1/2}).$$

$$\Rightarrow \|\vec{r}\,'(t)\| = \sqrt{(3\cos(3t))^2 + (-3\sin(3t))^2 + (3t^{1/2})^2}$$

$$= \sqrt{9(\cos^2 3t + \sin^2 3t + t)}$$

$$= 3\sqrt{1+t}.$$

Hence $\ell(u) = \int_0^u 3\sqrt{1+t}\, dt \underset{\begin{array}{l} v = 1+t \\ dv = dt \end{array}}{=} 3\int_0^u v^{1/2}\, dv$

$$= 3\left(\frac{2}{3} v^{3/2} \Big|_0^u\right) = 2u^{3/2}.$$

$$\boxed{\ell(u) = \int_0^u \|\vec{r}\,'(t)\|\, dt \text{ is called the arc-length function.}}$$

Lecture 25: Product rules for inner and scalar products

Monday, November 21, 2016 9:23 AM

Before we move to the next topic let me mention two rules of differentiation for vector-valued functions that sometimes come in handy:

Product Rules.

1. $$\frac{d}{dt}\left(\vec{r}_1(t)\cdot\vec{r}_2(t)\right) = \vec{r}_1\,'(t)\cdot\vec{r}_2(t) + \vec{r}_1(t)\cdot\vec{r}_2\,'(t).$$

2. $$\frac{d}{dt}\left(\vec{r}_1(t)\times\vec{r}_2(t)\right) = \vec{r}_1\,'(t)\times\vec{r}_2(t) + \vec{r}_1(t)\times\vec{r}_2\,'(t).$$

Lecture 25: Speeding up/slowing down (extra reading)

Monday, November 21, 2016 9:33 AM

<u>Ex.</u> Having the velocity vector $\vec{v}(t_0)$ and the acceleration vector $\vec{a}(t_0)$, can we say if the particle is speeding up or slowing down?

<u>Solution</u>. <u>speeding up</u> means the speed function $s(t)$ is increasing at t_0; and <u>slowing down</u> means the speed function is decreasing at t_0. It is easier to work with $s(t)^2 = \|\vec{v}(t)\|^2 = \vec{v}(t) \cdot \vec{v}(t)$.

So we have to find out if $\left.\frac{d}{dt}(s(t)^2)\right|_{t=t_0}$ is positive or negative.

$$\left.\frac{d}{dt}(s(t)^2)\right|_{t=t_0} = \left.\frac{d}{dt}(\vec{v}(t) \cdot \vec{v}(t))\right|_{t=t_0}$$

$$= \vec{v}\,'(t_0) \cdot \vec{v}(t_0) + \vec{v}(t_0) \cdot \vec{v}\,'(t_0)$$

(product rule for inner product.)

$$= 2\,\vec{a}(t_0) \cdot \vec{v}(t_0).$$

So it speeds up if the angle between $\vec{a}(t_0)$ and $\vec{v}(t_0)$ is acute, and it slows down if this angle is obtus.