Lecture 23: Lagrange multiplier method

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In the previous lecture we saw

Lagrange multiplier method

Suppose f and g are "nice" functions (with continuous partial derivatives). If the maximum or the minimum of $f(x,y)$ constrained to $g(x,y)=c_0$ occurs at (x_0,y_0), then $\nabla f(x_0,y_0) = c\,\nabla g(x_0,y_0)$ for some constant c.

Today we will see a few examples.

Ex. Find the max. and the min. of $f(x,y) = 2x+3y$ subject to $x^2+y^2=4$.

Solution. First notice that, since $x^2+y^2=4$ (a circle) is a closed and bounded region and f is continuous, f has a max. and min..

Now by Lagrange multiplier method, if f has a max. or min. at (x_0,y_0), then

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$$\begin{cases} \nabla f(x_0,y_0) = c\,\nabla g(x_0,y_0) \\ g(x_0,y_0) = 4 \end{cases}$$, where $f(x,y) = 2x+3y$ and $g(x,y) = x^2+y^2$.

$\nabla f(x,y) = (2,3)$ and $\nabla g(x,y) = (2x, 2y)$.

So $\begin{cases} (2,3) = c(2x,2y) \\ x^2+y^2 = 4 \end{cases}$. Therefore $\begin{cases} x = \frac{1}{c}, \; y = \frac{3}{2c} \\ x^2+y^2 = 4. \end{cases}$ ⊛

Hence $\left(\frac{1}{c}\right)^2 + \left(\frac{3}{2c}\right)^2 = 4$, and so

$\left(1+\frac{9}{4}\right)\frac{1}{c^2} = 4$, which implies $c^2 = \frac{13}{16}$. Thus

$c = \pm\frac{\sqrt{13}}{4}$. By ⊛ we get

$$(x,y) = \left(\frac{4}{\sqrt{13}}, \frac{6}{\sqrt{13}}\right) \text{ or } \left(\frac{-4}{\sqrt{13}}, \frac{-6}{\sqrt{13}}\right).$$

- $f\left(\frac{4}{\sqrt{13}}, \frac{6}{\sqrt{13}}\right) = 2\times\frac{4}{\sqrt{13}} + 3\times\frac{6}{\sqrt{13}} = \frac{26}{\sqrt{13}} = 2\sqrt{13}$ max.

  $f\left(-\frac{4}{\sqrt{13}}, -\frac{6}{\sqrt{13}}\right) = -2\sqrt{13}$ min.

<u>Ex.</u> Find the closest point to the origin on the plane $2x+3y+4z=9$.

<u>Solution</u>. So we need to find minimum of $f(x,y,z) = x^2+y^2+z^2$

constrained to $g(x,y,z) = 2x+3y+4z = 9$.

Pick a point P_0 on $g(x,y,z) = 9$. So the min. occurs with in

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the ball centered at the origin with radius OP_0. Since this part of plane is closed and bounded, and f is continuous, f has a min.

By Lagrange multiplier method, if min. occurs at (x,y,z) then

$$\begin{cases} \nabla f(x,y,z) = c\,\nabla g(x,y,z) \\ g(x,y,z) = 9 \end{cases}$$

$\nabla f(x,y,z) = (2x, 2y, 2z)$ and $\nabla g(x,y,z) = (2,3,4)$

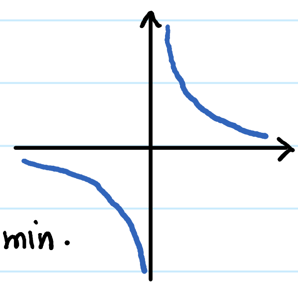
So $\begin{cases} (2x,2y,2z) = c\,(2,3,4). \\ 2x+3y+4z=9 \end{cases}$ Hence $\begin{cases} x=c,\ y=\frac{3}{2}c,\ z=2c \\ 2x+3y+4z=9. \end{cases}$

Therefore $2c+\frac{9}{2}c+8c=9$, which implies

$(2+\frac{9}{2}+8)\,c=9$. We get $c=\frac{18}{29}$, and so

$(x,y,z) = \left(\frac{18}{29}, \frac{27}{29}, \frac{36}{29}\right)$.

Ex. Find max. and min. values of x^2y+x+y constrained to $xy=4$.



Solution. As we can see, $xy=4$ is NOT a bounded region. So it is NOT clear whether x^2y+x+y has a max or min.

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Notice that, if (x,y) satisfy $xy=4$, then

$$x^2y+x+y = x(xy)+x+y = 4x+x+y = 5x+y.$$

So we have to find max. and min. of $5x+y$ subject to $xy=4$.

So $y=4/x$ and we have to optimize $5x+4/x$.

Notice $\lim_{x\to\infty} 5x+4/x = \infty$ and $\lim_{x\to-\infty} 5x+4/x = -\infty$.

So x^2y+x+y subject to $xy=4$ has neither max. nor min.

When the given region is NOT bounded or NOT closed, then we have to find out how the given function behaves as we go to infinity or towards the boundary with in the given region.

Ex. Find the max. and the min. of $f(x,y)=x^2y$ on the ellipse $4x^2+9y^2=36$.

Solution. An ellipse is closed and bounded, and f is continuous, so f has a max. and a min. By Lagrange multiplier method, if f has a max or min at (x_0,y_0), then we have

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$$\begin{cases} \nabla f(x_0,y_0) = c\nabla g(x_0,y_0) \\ g(x_0,y_0) = 36 \end{cases}$$, where $f(x,y) = x^2y$ and $g(x,y) = 4x^2+9y^2$.

Since $\nabla f = (2xy, x^2)$ and $\nabla g = (8x, 18y)$, we get

$$\begin{cases} (2xy, x^2) = c(8x, 18y) \\ 4x^2+9y^2 = 36 \end{cases}$$, which implies

$$\begin{cases} xy = 4cx & (1) \\ x^2 = 18cy & (2) \\ 4x^2+9y^2 = 36 & (3) \end{cases}$$

. $xy = 4cx$ implies that either $\underline{x=0}$ or $\underline{y=4c}$.

<u>Case 1.</u> $x \neq 0$.

In this case, $y = 4c$. So, by (2), $x^2 = (18c)(4c)$. We get $x = \pm 6\sqrt{2}\, c$. By (3),

$$36 = 4x^2+9y^2 = (4)(72)c^2 + (9)(16)c^2$$
$$= 36(8+4)c^2$$

Hence $c = \pm \frac{1}{\sqrt{12}} = \pm \frac{1}{2\sqrt{3}}$. Therefore the possible values of $f(x,y) = x^2y = (18)(4)c^2 \cdot 4c = (18)(16)c^3$

$$= \pm (18)(16)\frac{1}{(8)(3)\sqrt{3}} = \pm 4\sqrt{3}.$$

<u>Case 2.</u> $x=0$. Value of $f=0$.

Comparing these values we get: $\underline{\text{max.} = 4\sqrt{3}}$ and $\underline{\text{min.} = -4\sqrt{3}}$.