Lecture 22: Global maximum and minimum

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In the previous lecture we discussed what a bounded region and a closed region are.

Theorem Any continuous function has a global max and a global min on a closed and bounded region.

How can we find global max and global min ?

1. Find all the critical points of f inside D.
2. Find max and min on the boundary of D
3. Compare the values of f at the critical pts in D and the max and min of f on the boundary of D.

Ex. Find global max and global min of $f(x,y)=x^2+y^2-x-y$ in the disk $x^2+y^2\leq 1$.

Solution. Step 1. Find all critical points in $x^2+y^2<1$.

$\nabla f=(2x-1,\ 2y-1)=(0,0)$. So $x=\frac{1}{2}$ and $y=\frac{1}{2}$

$(1/2)^2+(1/2)^2=\frac{1}{2}<1$. So $(\frac{1}{2},\frac{1}{2})$ is a critical point in this region. And $f(\frac{1}{2},\frac{1}{2})=\frac{1}{2}-\frac{1}{2}-\frac{1}{2}=-\frac{1}{2}$.

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Step 2. Find max and min of $f(x,y)$ constrained to the circle $x^2+y^2=1$.

Let's parametrize the circle: $x = \cos\theta$, $y = \sin\theta$ and $0 \le \theta \le 2\pi$.

$$f(\cos\theta, \sin\theta) = \cos^2\theta + \sin^2\theta - \cos\theta - \sin\theta$$

$$= 1 - \cos\theta - \sin\theta$$

It is a single-variable function and you can use all the techniques from single-variable calculus:

Let $g(\theta) = 1 - \cos\theta - \sin\theta$. Then we have to find all θ's such that $g'(\theta) = 0$.

$g'(\theta) = \sin\theta - \cos\theta = 0$. So $\sin\theta = \cos\theta$. Therefore $\theta = \frac{\pi}{4}$ or $\pi + \frac{\pi}{4} = \frac{5\pi}{4}$.

Now we have to compare $g(\frac{\pi}{4})$, $g(\frac{5\pi}{4})$, $g(0)$, and $g(2\pi)$. We have $g(\frac{\pi}{4}) = 1 - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = 1 - \sqrt{2}$,
$g(\frac{5\pi}{4}) = 1 + \sqrt{2}$, $g(0) = g(2\pi) = 0$.

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So maximum of f on the circle occurs at $(\cos\frac{5\pi}{4}, \sin\frac{5\pi}{4})$

$= (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ and it is $f(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) = 1+\sqrt{2}$,

and min. of f on the circle occurs at $(\cos\frac{\pi}{4}, \sin\frac{\pi}{4})$

$= (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and it is $f(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = 1-\sqrt{2}$.

Step 3. Comparing $f(\frac{1}{2}, \frac{1}{2})$, $1+\sqrt{2}$, and $1-\sqrt{2}$.

$$f(\tfrac{1}{2}, \tfrac{1}{2}) = \frac{1}{4} + \frac{1}{4} - \frac{1}{2} - \frac{1}{2} = -\frac{1}{2}.$$

Clearly $1+\sqrt{2}$ is the largest value here.

$$-\frac{1}{2} < 1-\sqrt{2} \Leftarrow \sqrt{2} < 1+\frac{1}{2} \Leftarrow \sqrt{2} < \frac{3}{2} \Leftarrow 2 < \frac{9}{4}.$$

So global max $= f(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) = 1+\sqrt{2}$.

and global min $= f(\frac{1}{2}, \frac{1}{2}) = -\frac{1}{2}$.

- For step 2, there are two methods. The first method (as we did above) is to parametrize the boundary of the given region. Then one can find the max and min of single-variable function $f(\vec{r}(t))$ where $\vec{r}(t)$ is a parametrization of (part of) boundary of the given region. This method is particularly useful when the boundary consists of a few segments, e.g. it is a square or triangle.

The 2^{nd} method is <u>Lagrange multiplier method</u>. This method is applicable when the boundary is a level curve (or a level surface).

Lagrange multiplier method.

Problem. Find max and min of $f(x,y)$ constrained to $g(x,y)=c_0$.

Let's use the idea in the 1^{st} method and assume $\vec{r}(t)$ is a parametrization of $g(x,y)=c_0$. So max and min of f restricted to this curve occur at a critical point of the single-variable function $f(\vec{r}(t))$. This implies at the point $(x_0,y_0)=\vec{r}(t_0)$ we have

$$0=\frac{d}{dt}(f\circ\vec{r})(t_0) = \nabla f(x_0,y_0)\cdot \vec{r}\,'(t_0).$$

And so $\nabla f(x_0,y_0) \perp \vec{r}\,'(t_0)$.

At the same time $\nabla g(x_0,y_0)$ is perpendicular to the

the level curve $g(x,y)=c_0$. So $\vec{r}\,'(t_0)$ (which is parallel to the tangent line of $g(x,y)=c_0$ at (x_0,y_0)) is perpendicular to $\nabla g(x_0,y_0)$.

Since both $\nabla f(x_0,y_0)$ and $\nabla g(x_0,y_0)$ are perpendicular to $\vec{r}\,'(t_0)$, we have

$$\nabla f(x_0,y_0) = c\, \nabla g(x_0,y_0)$$

(A similar argument works for 3 variable functions.)

Because of the <u>multiplier</u> c, it is called Lagrange multiplier method (the reason for the Lagrange part of this name should be clear!)