Lecture 17: Chain rule, gradient, directional derivative

Monday, October 31, 2016 8:40 AM

At the end of the previous lecture we were discovering <u>chain rule</u>:

Let $\vec{r}(t)=(x(t),y(t))$; $z=f(x,y)$. Then what is

$$\left(\frac{d}{dt}(f\circ\vec{r})\right)(t_0) \ ?$$

We assumed f is differentiable at $\vec{r}(t_0)$ and approximated f with an affine function:

$$f(x,y) \approx f(\vec{r}(t_0)) + f_x(\vec{r}(t_0))\,(x-x(t_0)) + f_y(\vec{r}(t_0))\,(y-y(t_0))$$

So
$$f(\vec{r}(t_0+\Delta t))-f(\vec{r}(t_0)) \approx f_x(\vec{r}(t_0))\,(x(t_0+\Delta t)-x(t_0)) + f_y(\vec{r}(t_0))\,(y(t_0+\Delta t)-y(t_0))$$

Therefore
$$\left(\frac{d}{dt}(f\circ\vec{r})\right)(t_0) = \lim_{\Delta t\to 0}\frac{f(\vec{r}(t_0+\Delta t))-f(\vec{r}(t_0))}{\Delta t}$$

$$= f_x(\vec{r}(t_0))\lim_{\Delta t\to 0}\frac{x(t_0+\Delta t)-x(t_0)}{\Delta t} + f_y(\vec{r}(t_0))\lim_{\Delta t\to 0}\frac{y(t_0+\Delta t)-y(t_0)}{\Delta t}$$

$$= f_x(\vec{r}(t_0))\,x'(t_0) + f_y(\vec{r}(t_0))\,y'(t_0)$$

$$\boxed{\left(\frac{d}{dt}(f\circ\vec{r})\right)(t_0) = \frac{\partial f}{\partial x}(\vec{r}(t_0))\,x'(t_0) + \frac{\partial f}{\partial y}(\vec{r}(t_0))\,y'(t_0)}$$

Lecture 17: Chain rule, gradient

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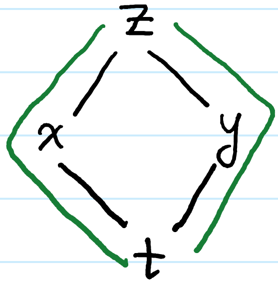
How can we remember the symbols and formula?

We have term associated to each path from z to the bottom row, (here t).

z depends on x and y

x and y depend on t



$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

d is used if there is only one variable in that row.

∂ is used if there are more than one variables in that row

Another way of viewing this equation is as follows:

$$\left(\frac{d}{dt}(f \circ \vec{r})\right)(t_0) = \underbrace{\left(\frac{\partial f}{\partial x}(\vec{r}(t_0)), \frac{\partial f}{\partial y}(\vec{r}(t_0))\right)} \cdot \underbrace{(x'(t_0), y'(t_0))}_{\vec{r}\,'(t_0)}$$

This is called gradient of f

(as you have seen by now) and

it is denoted by ∇f (the nabla symbol)

So we have $\boxed{\frac{d}{dt} f(r(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)}$

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Before we see a numerical example, let me mention that the above formulas work for 3 variable functions and 3-dimensional vector-valued functions as well:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

$$\begin{tikzcd} & w \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & y \arrow[d, dash] & z \arrow[dl, dash] \\ & t & \end{tikzcd}$$

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

where $\nabla f(\vec{v}) = \left(\frac{\partial f}{\partial x}(\vec{v}), \frac{\partial f}{\partial y}(\vec{v}), \frac{\partial f}{\partial z}(\vec{v})\right)$.

<u>Ex.</u> Let $f(x,y) = 5 - x^2 + 9y^2$ and $\vec{r}(t) = (\cos t, \sin t)$.

Find $\nabla f(x,y)$, $\vec{r}\,'(t)$, and $\frac{d}{dt}(f\circ\vec{r})(\pi)$.

<u>Solution</u>. $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)$

$= (-2x, 18y)$.

$\vec{r}\,'(t) = (-\sin t, \cos t)$.

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

$$= (-2\cos t, 18\sin t) \cdot (-\sin t, \cos t)$$

$$= 2\cos t \cdot \sin t + 18 \sin t \cdot \cos t$$

$$= 20 \cos t \cdot \sin t.$$

$$\frac{d}{dt}(f\circ\vec{r})(\pi) = (20)(\cos\pi)(\sin\pi) = 0.$$

Lecture 17: Gradient, chain rule

Wednesday, November 2, 2016 1:16 AM

In the previous example we could first compute $f\circ\vec{r}$ and then compute its derivative:

$$(f\circ\vec{r})(t) = 5 - \cos^2 t + 9\sin^2 t$$

So $\frac{d}{dt}(f\circ\vec{r})(t) = +2\sin t\cdot\cos t + 18\cos t\cdot\sin t$

$$= 20\sin t\cdot\cos t.$$

So why did we bother getting $\frac{d}{dt}(f\circ\vec{r})(t_0) = \nabla f(\vec{r}(t_0))\cdot\vec{r}\,'(t_0)$?

For various reasons:

① In practice it is not easy to find a formula for a function, but we can design some experiments to compute ∇f at certain points (x_0, y_0). Then this can help us to understand $\frac{d}{dt}(f\circ\vec{r})(t_0)$ for a lot of curves $\vec{r}(t)$.

①' Suppose you are told $\vec{r}\,'(t_0)$ and $\nabla f(\vec{r}(t_0))$.

Without knowing what f or $\vec{r}$ are, you can compute $\frac{d}{dt}(f\circ\vec{r})(t_0)$.

② Helps us to get a much better understanding of ∇f, which in turn helps us to find tangent planes of a lot of surfaces.

Lecture 17: Gradient, level curves and surfaces

Wednesday, November 2, 2016 1:27 AM

Suppose $\vec{r}(t)$ is a parametrization of a level curve $f(x,y) = c_0$. So, for any t, $f(\vec{r}(t)) = c_0$.

Hence $\frac{d}{dt}(f(\vec{r}(t))) = 0$, and so by chain rule:

$$0 = \frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t),$$

which implies $\nabla f(\vec{r}(t)) \perp \vec{r}\,'(t)$. Since $\vec{r}\,'(t)$ is parallel to the tangent line of this curve at $\vec{r}(t)$, we get

$$\boxed{\nabla f(x_0,y_0) \perp \text{ level curve } f(x,y) = f(x_0,y_0).}$$

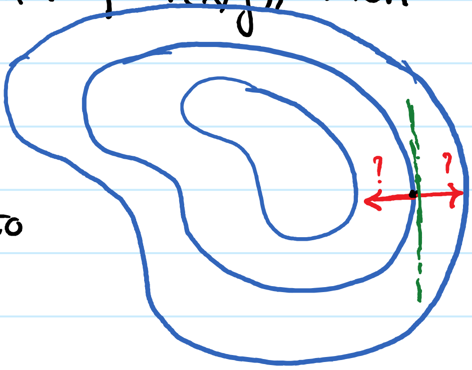
Similarly for 3 variable functions we get

$$\boxed{\nabla f(x_0,y_0,z_0) \perp \text{ level surface } f(x,y,z) = f(x_0,y_0,z_0)}$$

which implies

$$\boxed{\begin{array}{l}\nabla f(x_0,y_0,z_0) \text{ is a normal vector of the tangent plane of the} \\ \text{surface } f(x,y,z) = f(x_0,y_0,z_0) \text{ at the point } (x_0,y_0,z_0).\end{array}}$$

So if the following is the contour diagram of $f(x,y)$, then $\nabla f(x_0,y_0)$ is perpendicular to the green line. And so for its direction there are only two possibilities. Later we will see ∇f goes to the direction where f increases.



Lecture 17: Tangent plane

Thursday, November 3, 2016 12:45 AM

<u>Ex.</u> Find an equation of the tangent plane of $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$ at the point (x_0, y_0, z_0).

<u>Solution</u>. Let $f(x,y,z)= \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$. So we want to find equation of a tangent plane of the level surface $f(x,y,z)=1$. So $\nabla f(x_0,y_0,z_0)$ is a normal vector of this plane.

$$\nabla f(x_0,y_0,z_0) = (f_x(x_0,y_0,z_0), f_y(x_0,y_0,z_0), f_z(x_0,y_0,z_0))$$

$$= \left(\frac{2x_0}{a^2}, \frac{2y_0}{b^2}, \frac{2z_0}{c^2}\right).$$

So equation of the tangent plane at (x_0,y_0,z_0) is

$$\frac{2x_0}{a^2}(x-x_0)+\frac{2y_0}{b^2}(y-y_0)+\frac{2z_0}{c^2}(z-z_0)=0.$$

So after cancelling 2 and bringing constants to the right hand side we get:

$$\frac{x_0 x}{a^2}+\frac{y_0 y}{b^2}+\frac{z_0 z}{c^2} = \frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}.$$

Since (x_0,y_0,z_0) is a point on this ellipsoid, we have

$$\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}=1.$$

So we get $\frac{x_0 x}{a^2}+\frac{y_0 y}{b^2}+\frac{z_0 z}{c^2}=1$.