Lecture 15: Vector-valued functions

Friday, October 28, 2016 9:25 AM

So far we have been working with real-valued functions, i.e. functions whose values are real numbers. Now we would like to study functions whose values are vectors (either with two or three variables).

We have seen such functions when we parametrized a line parallel to a vector $\vec{v}_0=(a,b,c)$ and passing through (x_0,y_0,z_0):

$$\vec{l}(t) = (x_0, y_0, z_0) + t\,\vec{v}_0 = (x_0+at,\, y_0+bt,\, z_0+ct).$$

A (single-variable) vector-valued function is of the form:

$$\vec{r}(t) = (x(t), y(t), z(t)).$$

As we saw, vector-valued functions are essential to parametrize curves. [You will be using this in your next Calculus course to compute line integrals; which is needed to find the work done by a force for moving a particle along a curve.]

Another application of vector-valued functions is in studying the behavior of a moving particle: its position, velocity, speed, and acceleration.

Lecture 15: Limit of a vector-valued function

Saturday, October 29, 2016 12:02 AM

We would like to do calculus with vector-valued functions:

limit, derivative, etc.

Limit is fairly easy: if $\vec{r}(t) = (x(t), y(t), z(t))$, then

$$\boxed{\lim_{t\to t_0} \vec{r}(t) = \left(\lim_{t\to t_0} x(t), \lim_{t\to t_0} y(t), \lim_{t\to t_0} z(t)\right).}$$

So essentially one has to compute three limits of single-variable functions.

<u>Ex.</u> Find $\lim_{t\to 0}\left(\frac{e^t-1}{t}, \frac{\sin t}{t}, t\right)$.

<u>Solution</u>.

$$\lim_{t\to 0}\left(\frac{e^t-1}{t}, \frac{\sin t}{t}, t\right) = \left(\lim_{t\to 0}\frac{e^t-1}{t}, \lim_{t\to 0}\frac{\sin t}{t}, \lim_{t\to 0} t\right)$$

$$\underset{\text{l'Hôpital's rule}}{=} \left(\lim_{t\to 0}\frac{e^t}{1}, \lim_{t\to 0}\frac{\cos t}{1}, 0\right)$$

$$= (1, 1, 0).$$

What is derivative $\vec{r}\,'(t)$ of $\vec{r}(t) = (x(t), y(t), z(t))$?

We use the usual definition: $\vec{r}\,'(t)$ is the rate of change of $\vec{r}(t)$ with respect to t. I.e.

$$\vec{r}\,'(t) = \lim_{\Delta t\to 0}\frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

Lecture 15: Derivative of a vector-valued function

Saturday, October 29, 2016 12:14 AM

So $$\vec{r}\,'(t) = \lim_{\Delta t \to 0} \frac{\left(x(t+\Delta t) - x(t),\ y(t+\Delta t) - y(t),\ z(t+\Delta t) - z(t)\right)}{\Delta t}$$

$$= \left(\lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t},\ \lim_{\Delta t \to 0} \frac{y(t+\Delta t) - y(t)}{\Delta t},\ \lim_{\Delta t \to 0} \frac{z(t+\Delta t) - z(t)}{\Delta t}\right)$$

$$= (x'(t), y'(t), z'(t)).$$

Hence, for $\vec{r}(t) = (x(t), y(t), z(t))$, we have $\vec{r}\,'(t) = (x'(t), y'(t), z'(t))$

__Ex__. Suppose $\vec{r}(t) = (e^t, t^2, \ln t)$. Find $\vec{r}\,'(1)$.

__Solution__. $\vec{r}\,'(t) = (e^t, 2t, 1/t)$. So $\vec{r}\,'(1) = (e, 2, 1)$.

Moving particle.

Suppose a moving particle is at $\vec{r}(t) = (x(t), y(t), z(t))$ after t seconds. How can we compute its velocity, speed, and accelleration?

Velocity is the rate of change of the positional vector of the particle with respect to time: $\vec{v}(t) = \vec{r}(t)$.

Similarly, accleration is the rate of change of the velocity with respect to time: $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$

with respect to time: $\boxed{\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)}$

Lecture 15: Velocity, acceleration, speed

Saturday, October 29, 2016 12:34 AM

Speed is the length of velocity: $\boxed{s(t) = \|\vec{v}(t)\| = \|\vec{r}\,'(t)\|.}$

Ex. Suppose a moving particle is at the point $\vec{r}(t) = (2\cos t, 2\sin t, t)$ at time t.

(a) Find its velocity at $t = \pi$

(b) Find its speed at t.

(c) Find its acceleration at $t = \pi$.

Solution (a) $\vec{v}(t) = \vec{r}\,'(t) = (-2\sin t, 2\cos t, 1)$.

So $\vec{v}(\pi) = (-2\sin\pi, 2\cos\pi, 1) = (0, -2, 1)$.

(b) $s(t) = \|\vec{v}(t)\| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (1)^2}$

$$= \sqrt{4(\sin^2 t + \cos^2 t) + 1} = \sqrt{5}.$$

(c) $\vec{a}(t) = \vec{v}\,'(t) = (-2\cos t, -2\sin t, 0)$

So $\vec{a}(\pi) = (-2\cos\pi, -2\sin\pi, 0)$

$$= (2, 0, 0).$$

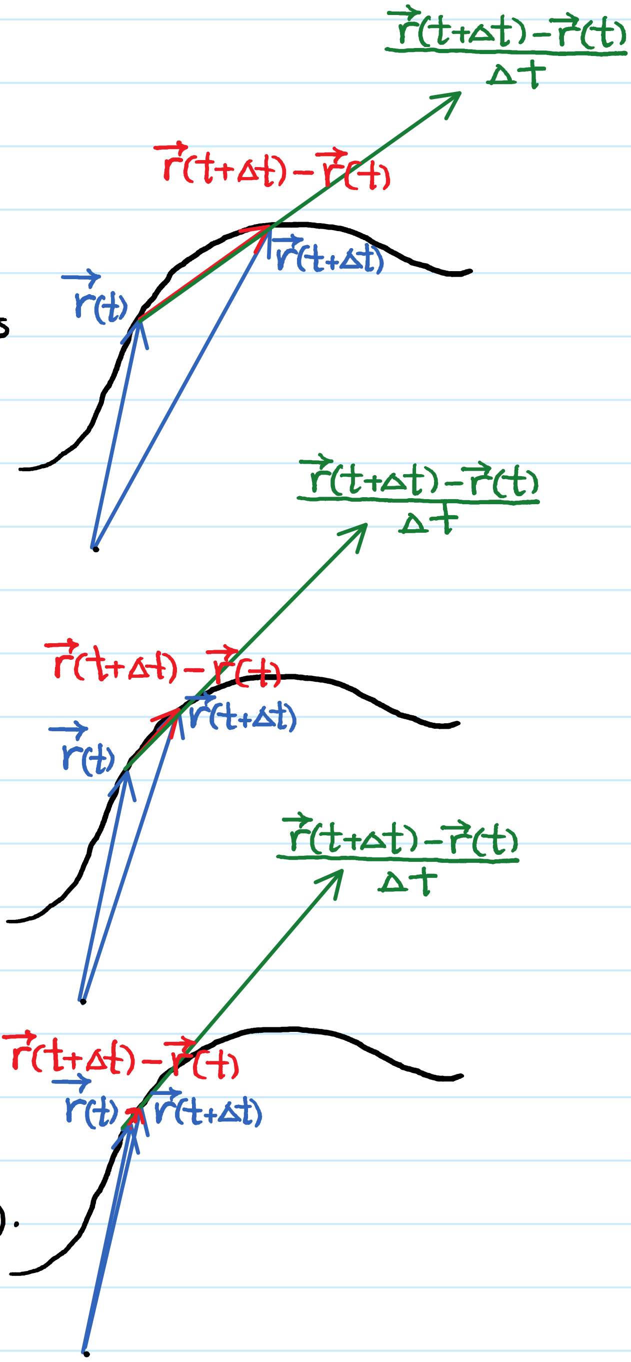
Remark. Constant speed does <u>not</u> imply that the acceleration is zero as we saw in the above example. Constant velocity implies that the acceleration is zero.

Lecture 15: Tangent line

Saturday, October 29, 2016 12:50 AM

In single-variable calculus, we know how to interpret $f'(t)$: it is slope of the tangent line.

How can we visualize $\vec{r}\,'(t)$?



As you can, $\frac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t}$ converges to a vector which is parallel to the tangent line of the curve parametrized by $\vec{r}$ at the point $\vec{r}(t)$. (It might be $\vec{0}$.)

So it can help us to get a parametrization of the tangent line of $\vec{r}(t)$ at the point $\vec{r}(t_0)$.

$$\boxed{\ell(t)=\vec{r}(t_0)+t\,\vec{r}\,'(t_0) \text{ is a parametrization of the tangent line of the curve given by } \vec{r}(t) \text{ at the point } \vec{r}(t_0).}$$

Lecture 15: Tangent line

Sunday, October 30, 2016 10:29 AM

Ex. Parametrize the tangent line of $\vec{r}(t) = (t, t^2, t^3)$ at the point $(1,1,1)$.

Solution. A parametrization of the tangent line at $\vec{r}(t_0)$ is $\vec{l}(t) = \vec{r}(t_0) + t\,\vec{r}\,'(t_0)$. So we need to find t_0 and $\vec{r}\,'(t)$.

Since $\vec{r}(t_0)$ is supposed to be $(1,1,1)$, we get

$$(1,1,1) = (t_0, t_0^2, t_0^3). \quad \text{So } t_0 = 1.$$

$\vec{r}\,'(t) = (1, 2t, 3t^2)$. Hence

$$\vec{l}(t) = \vec{r}(1) + t\,\vec{r}\,'(1) = (1,1,1) + t(1,2,3)$$

$$= (t+1, 2t+1, 3t+1).$$