Lecture 12: Limit

Wednesday, October 19, 2016 8:41 AM

In the previous lecture we mentioned how we can use polar coordinates to find a limit of a two-variable function:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} f(r\cos\theta(r), r\sin\theta(r))$$

Here $\theta(r)$ is an unknown function.

Ex. Determine if the following limit exist and, if it does, find the limit.
$$\lim_{(x,y)\to(0,0)} \frac{x^{0.5}\, y^{1.75}}{x^2+y^2}.$$

> **Remark.** The same solution shows that
> $$\lim_{(x,y)\to(0,0)} \frac{|x|^a\,|y|^b}{x^2+y^2} = 0$$
> if a and b are non-negative and $a+b>2$.

Solution. We use polar coordinates:

$$\lim_{(x,y)\to(0,0)} \frac{|x|^{0.5}\,|y|^{1.75}}{x^2+y^2} = \lim_{r\to 0} \frac{(|r\cos\theta(r)|)^{0.5}(|r\sin\theta(r)|)^{1.75}}{r^2\cos^2\theta(r)+r^2\sin^2\theta(r)}$$

$$= \lim_{r\to 0} \frac{r^{2.25}\,|\cos\theta(r)|^{0.5}\,|\sin\theta(r)|^{1.75}}{r^2}$$

$$= \lim_{r\to 0} r^{0.25}\,|\cos\theta(r)|^{0.5}\,|\sin\theta(r)|^{1.75}.$$

Since $\lim_{r\to 0} r^{0.25} = 0$ and $0 \le r^{0.25}|\cos\theta(r)|^{0.5}|\sin\theta(r)|^{1.75} \le r^{0.25}$

by squeeze theorem we have $\lim_{r\to 0} r^{0.25}|\cos\theta(r)|^{0.5}|\sin\theta(r)|^{1.75} = 0.$

Lecture 12: partial derivatives

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What is the rate of change of $f(x,y)$ with respect to x and y?

I.e.. How fast is $f(x,y)$ changing as $\underline{x}$ or $\underline{y}$ varies?

- Does $f(x,y)$ increase as x increases?

Ex. Let $f(x,y)=x-y$. At $(1,1)$ does f increase as x increases? What if y increase?

Answer. It increases as x increases. And decreases as y increases.

To understand "how fast it changes", its rate of change, we need to find:

$$\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x} \quad \text{and} \quad \lim_{\Delta y\to 0}\frac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}$$

rate of change with respect to x | rate of change with respect to y.

By fixing $y=y_0$ and viewing $f(x,y_0)$ as a single-variable function, we see that $\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$ is the derivative of $f(x,y_0)$ with respect to x. It is called partial derivative of f with respect to x. It is denoted by $f_x(x_0,y_0)$ or $\frac{\partial f}{\partial x}(x_0,y_0)$

Lecture 12: Partial derivatives

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Similarly, rate of change of $f(x,y)$ with respect to y, is the derivative of $f(x_0, y)$ with respect to y. It is called partial derivative of $f(x,y)$ with respect to y. It is denoted by $f_y(x_0, y_0)$ or $\frac{\partial f}{\partial y}(x_0, y_0)$.

Ex. Find partial derivatives of $f(x,y) = x - yx^2$.

Solution. $f_x = 1 - 2xy$, and $f_y = -x^2$.

Ex. Find partial derivatives of $f(x,y) = \sin\left(\frac{x}{y}\right)$.

Solution. $f_x = \frac{1}{y}\cos\left(\frac{x}{y}\right)$ [we viewed f as a function of x, i.e. y is viewed as a constant. Then we used chain-rule for single-variable functions. Derivative of $\frac{x}{y_0}$ is $\frac{1}{y_0}x$.]

$f_y = \frac{-x}{y^2}\cos\left(\frac{x}{y}\right)$.

Ex. Find partial derivatives of $f(x,y) = \sqrt{x^2+y^2}$.

Solution. $f_x = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$ [Again we used chain-rule for single-variable functions.]

$f_y = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}$.

Lecture 12: Geometric view of partial derivatives

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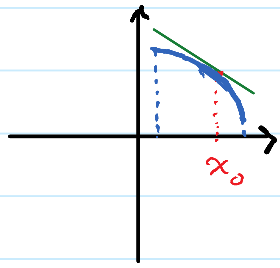
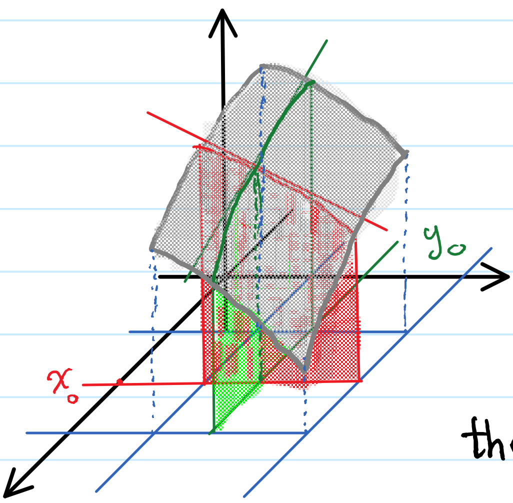
In the single-variable case derivative of f at x_0 gives as the slope of the tangent line of the graph of f.

Hence $f_x(x_0,y_0)$ is the slope of the tangent line of graph of $f(x,y_0)$ viewed as a function of f.



We can view graph of $f(x,y_0)$ as the curve of intersection of $z=f(x,y)$ and the plane $y=y_0$.



The grean plane is $y=y_0$ and its intersection with $z=f(x,y)$ is the graph of $f(x,y_0)$

so $f_x(x_0,y_0)$ is the slope of the green line.

Similarly the red plane is $x=x_0$ and its intersection with $z=f(x,y)$ is the graph of $f(x_0,y)$. So $f_y(x_0,y_0)$ is the slope of the red line.

Lecture 12: Preparation for tangent plane

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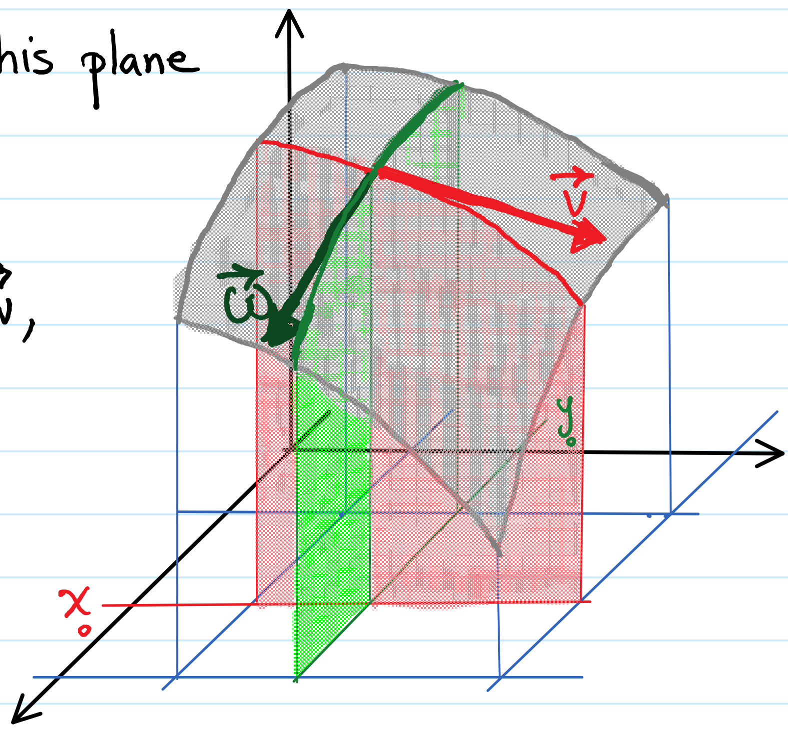
Graph of two-variable function is a "two dimensional" object. So tangent lines are NOT good enough to understand graph of $f(x,y)$. In this case we should seek for a tangent plane.

If there is a tangent plane of $z=f(x,y)$ at $(x_0, y_0, f(x_0,y_0))$, then it should contain the tangent lines of the "red" and the "green" curves. So it is parallel to the direction of these tangent lines: $\vec{v}$ and $\vec{w}$. Hence to find a normal vector of this plane we need to find $\vec{v}$ and $\vec{w}$, and then $\vec{v} \times \vec{w}$.



This will be done in the next lecture.