Lecture 10: Level surfaces

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In the previous lecture we saw how contour diagram and graph of a two-variable function can help us to visualize a two-variable function.

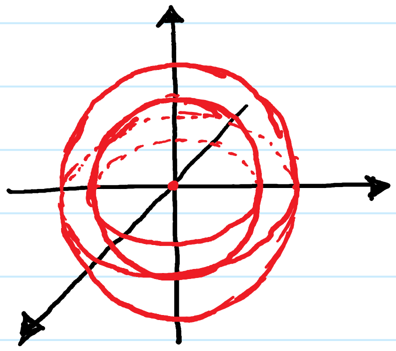
We cannot sketch graph of a <u>three-variable</u> function, but we can use <u>level surfaces</u>: we visualize what points in the domain have the same value of $f(x,y,z)$.

<u>Ex.</u> Sketch level surfaces of $f(x,y,z) = x^2+y^2+z^2$.

<u>Solution</u>. $f(x,y,z)=0$ is just $(0,0,0)$

$f(x,y,z)=1 \Rightarrow x^2+y^2+z^2=1$

(it says the distance from the origin is 1.)



$\Rightarrow$ sphere centered at the origin with radius 1.

$f(x,y,z) = 2 \Rightarrow x^2+y^2+z^2 = 2$ sphere centered at $(0,0,0)$ with radius 1.

<u>Ex.</u> Sketch level surfaces of $f(x,y,z) = x^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{3}\right)^2$.

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Solution. $f(x,y,z)=0 \Rightarrow x^2+\frac{y^2}{4}+\frac{z^2}{9}=0 \Rightarrow (0,0,0)$

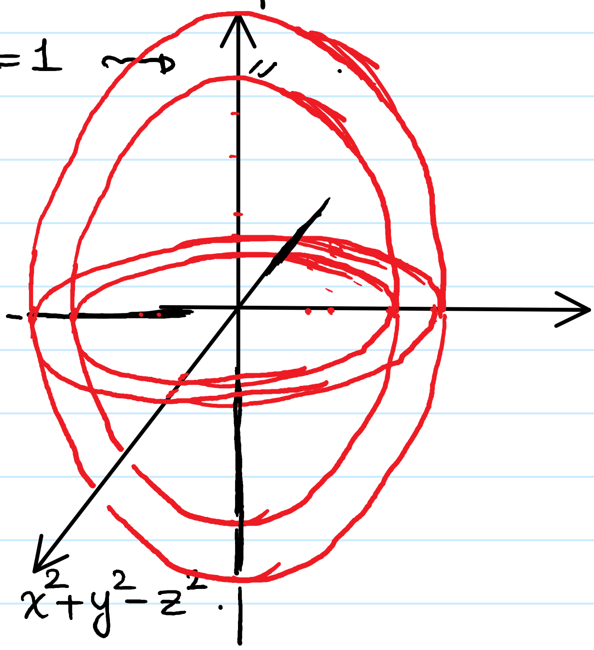
$f(x,y,z)=1 \Rightarrow x^2+\frac{y^2}{4}+\frac{z^2}{9}=1$.

Slice it by planes parallel to xy-plane, yz-plane, xz-plane.

xy-plane $\rightsquigarrow z=0 \rightsquigarrow x^2+\frac{y^2}{4}=1 \rightsquigarrow$ ellipse.

yz-plane $\rightsquigarrow x=0 \rightsquigarrow \frac{y^2}{4}+\frac{z^2}{9}=1 \rightsquigarrow$

(It is called an ellipsoid.)

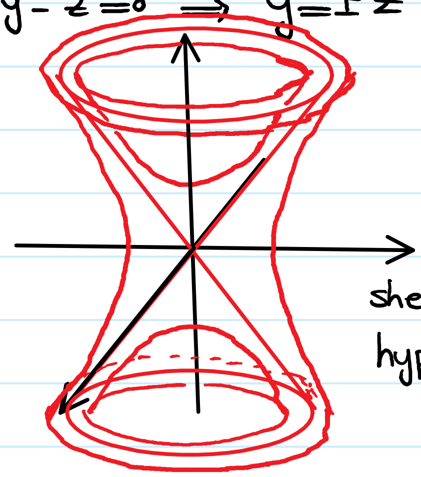


Ex. Sketch level surfaces of $f(x,y,z)=x^2+y^2-z^2$.

Solution. $f(x,y,z)=0 \rightsquigarrow 0=x^2+y^2-z^2$

Intersect it with the yz-plane: $y^2-z^2=0 \Rightarrow y=\pm z$.

Intersect it with $z=\pm 1$: $x^2+y^2=1$



one-sheeted hyperboloid.

$f(x,y,z)=1 \rightsquigarrow 1=x^2+y^2-z^2$

intersect it with $x=0$: $y^2-z^2=1$

intersect it with $z=\pm 1$: $x^2+y^2=2$

$f(x,y,z)=-1 \rightsquigarrow -1=x^2+y^2-z$ using the above method, you get a two-sheeted hyperboloid.

Lecture 10: Limits

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Limits of multi-variable functions is similar to the single-variable case, but one has to be extra careful.

Ex. Find $\lim_{(x,y)\to(0,0)} \frac{\cos(x^2+y^2)}{1+x+y}$.

Solution. Since this is a "nice" function and $\frac{\cos(0)}{1+0} = 1$,

$$\lim_{(x,y)\to(0,0)} \frac{\cos(x^2+y^2)}{1+x+y} = 1.$$

Rule 1. When the function is "nice" and one plugs in and no problem arises, that is your answer.

Ex. Find $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$.

Solution. Since $\lim_{(x,y)\to(0,0)} x^2+y^2=0$, we have to understand the behavior of $\frac{\sin(t)}{t}$ as t approaches to 0. Since

$\lim_{t\to 0} \frac{\sin t}{t} = \lim_{t\to 0} \frac{\cos t}{1} = 1$, we have

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = 1.$$

Rule 2. $\lim_{(x,y)\to(a,b)} f(x,y) = t_0$ and $\lim_{t\to t_0} g(t) = L$; then

$$\lim_{(x,y)\to(a,b)} g(f(x,y)) = L.$$

Lecture 10: Limits

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The 2^{nd} rule is particularly useful when we are interested in limit of a multi-variable function which is a composite of a simpler multi-variable function and a single-variable function.

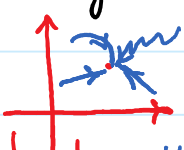
Ex. Find $\lim_{(x,y)\to(0,0)} \dfrac{\cos(\sqrt{x^2+y^2})-1}{x^2+y^2}$.

Solution. Let $t=\sqrt{x^2+y^2}$. Then we have to understand $\lim_{t\to 0} \dfrac{\cos t -1}{t^2}$ (in fact it is enough to understand $\lim_{t\to 0^+} \dfrac{\cos t -1}{t^2}$ as $\sqrt{x^2+y^2}\geq 0$.)

$$\lim_{t\to 0} \frac{\cos t-1}{t^2} = \lim_{t\to 0} \frac{-\sin t}{2t} = \lim_{t\to 0} \frac{-\cos t}{2} = -\frac{1}{2}.$$

So $\lim_{(x,y)\to(0,0)} \dfrac{\cos(\sqrt{x^2+y^2})-1}{x^2+y^2} = -\dfrac{1}{2}$.

When Rule 1 and Rule 2 do not apply, then you should ask yourself maybe the limit does NOT exist! The key conceptual idea is the following: if $\lim_{(x,y)\to(a,b)} f(x,y)=L$, then no matter how one



approaches (a,b) the value of f should approach towards L. Hence if along two different curves value of f approaches to two different numbers, $\lim_{(x,y)\to(a,b)} f(x,y)$ does NOT exist.

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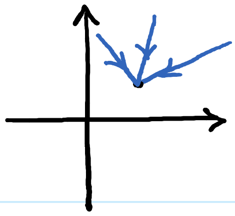

The easiest curves are lines:

Rule 3. Approach (a,b) along lines: $y-b=k(x-a)$

and see if $\lim_{x\to a} f(x, k(x-a)+b)$ depends on k or not.

If it depends on k, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does NOT exist

(If it does NOT depend on k, one cannot conclude anything!)

Ex. Determine whether the following limit exist or not.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}.$$

Solution. [Clearly we cannot use Rule 1 and Rule 2.]

Let's approach $(0,0)$ along lines: $y=kx$.

$$\lim_{x\to 0} \frac{(x)(kx)}{x^2+(kx)^2} = \lim_{x\to 0} \frac{k\,x^2}{x^2+k^2x^2} = \lim_{x\to 0} \frac{k}{1+k^2} = \frac{k}{1+k^2}.$$

It depends on k so $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does NOT exist.