Lecture 8: Volume of a parallelepiped; planes

Monday, October 10, 2016 12:10 AM

In the previous lecture we saw that

the volume of the parallelepiped spanned by $\vec{v}, \vec{w}$, and $\vec{u}$ $= |(\vec{v} \times \vec{w}) \cdot \vec{u}| = \left|\det \begin{bmatrix} \vec{v} \\ \vec{w} \\ \vec{u} \end{bmatrix}\right|$.

Ex. Find the volume of the parallelepiped spanned by

$$\vec{v} = (1,0,1), \ \vec{w} = (0,2,1) \text{ and } \vec{u} = (1,1,0).$$

Solution. volume $= \left|\det \begin{bmatrix} \vec{v} \\ \vec{w} \\ \vec{u} \end{bmatrix}\right| = \left|\det \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}\right|$

$$= \left| 1 \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} - 0 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + 1 \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} \right|$$

$$= |-1-2| = 3.$$

Planes.

A plane can be described in the following ways:

① passing through a point $P_0 = (x_0, y_0, z_0)$;

perpendicular to the vector $\vec{n}_0 = (a,b,c)$.

($\vec{n}_0$ is called a normal vector of this plane.)

② passing through a point P_0

parallel to two vectors $\vec{v}$ and $\vec{\omega}$

(with the assumption that $\vec{v}$ and $\vec{\omega}$ are NOT parallel.)

Lecture 8: Equations of planes

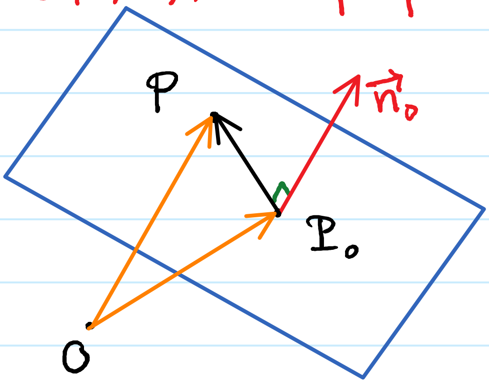
Monday, October 10, 2016 12:24 AM

③ Passing through three points A, B, C

(with the assumption that they are NOT collinear.)

④ Parallel to a plane and passes through a point.

Case ① Passes through $P_0 = (x_0, y_0, z_0)$
with normal vector $\vec{n_0} = (a, b, c)$, i.e. perpendicular to $\vec{n_0}$.



For any point $P = (x, y, z)$ on this plane $\vec{n_0} \perp \vec{P_0P}$. So

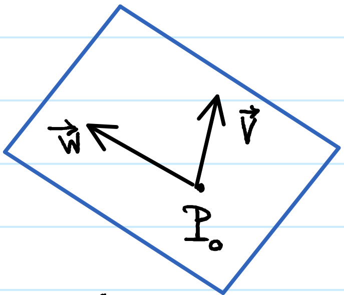
$$0 = \vec{n_0} \cdot \vec{P_0P} = \vec{n_0} \cdot (\vec{OP} - \vec{OP_0})$$

$$\vec{n_0} \cdot \vec{OP} = \vec{n_0} \cdot \vec{OP_0}.$$

Hence

$$\boxed{ax + by + cz = ax_0 + by_0 + cz_0.}$$
equation of the plane passing through (x_0, y_0, z_0); perpendicular to (a, b, c)

Case ② Passes through $P_0 = (x_0, y_0, z_0)$
Parallel to two vectors $\vec{v}$ and $\vec{w}$.



In order to use Case 1, we need to find a normal vector $\vec{n_0}$. So we need to find a vector which is perpendicular to $\vec{v}$ and $\vec{w}$

Lecture 8: Equation of a plane

Monday, October 10, 2016 12:45 AM

As we have seen, $\vec{v}\times\vec{w}$ is perpendicular to $\vec{v}$ and $\vec{w}$. So it is a normal vector of a plane parallel to $\vec{v}$ and $\vec{w}$. Hence

$$\boxed{\begin{array}{l}\text{To find equation of a plane}\\ \text{Passing through } P_0 \text{ and}\\ \text{Parallel to } \vec{v} \text{ and } \vec{w}\\ \text{Use } \vec{n}_0=\vec{v}\times\vec{w} \text{ is a normal vector}\\ \text{and use } \vec{n}_0\cdot\vec{OP}=\vec{n}_0\cdot\vec{OP_0}.\end{array}}$$

Case (3) Passing through A, B, C.

This plane is oing to be parallel to $\vec{AB}$ and $\vec{AC}$.

So as in Case (2) we get that

$$\boxed{\vec{n}_0=\vec{AB}\times\vec{AC} \text{ is a normal vector of this plane}}$$

And then we can make use of $\vec{n}_0\cdot\vec{OP}=\vec{n}_0\cdot\vec{OA}$.

Case (4) Parallel to $ax+by+cz=d$ and passing through P_0.

$\vec{n}_0=(a,b,c)$ is a normal vector of $ax+by+cz=d$,

So $\boxed{\vec{n}_0 \text{ is also perpendicular to the desired plane.}}$ Hence we can make use of $\vec{n}_0\cdot\vec{OP}=\vec{n}_0\cdot\vec{OP_0}$.

Lecture 8: Equation of a plane

Monday, October 10, 2016 12:58 AM

As you can see Case ①: a point P_0 and a normal vector $\vec{n}_0$ is the most important case. In the rest of the cases one would try to reduce to Case ①, i.e. find a normal vector.

Ex. Find an equation of a plane which passes through

$A = (1,0,1)$, $B = (0,2,1)$, and $C = (1,2,0)$.

Solution. $\vec{n}_0 = \vec{AB} \times \vec{AC}$ is a normal vector

$$\vec{AB} = (0,2,1) - (1,0,1) = (-1,2,0).$$

$$\vec{AC} = (1,2,0) - (1,0,1) = (0,2,-1).$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 0 \\ 0 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 2 & -1 \end{vmatrix} \vec{i} - \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} \vec{j} + \begin{vmatrix} -1 & 2 \\ 0 & 2 \end{vmatrix} \vec{k}$$

$$= -2\vec{i} - \vec{j} - 2\vec{k}.$$

So $-2x - y - 2z = (-2)(1) + (-1)(0) + (-2)(1)$

$= -2 - 2 = -4$

is an equation of this plane:

$$2x + y + 2z = 4.$$

Ex. Find an equation of the plane passing through $P_0 = (1,1,1)$ and parallel to $x - y + z = 3$.

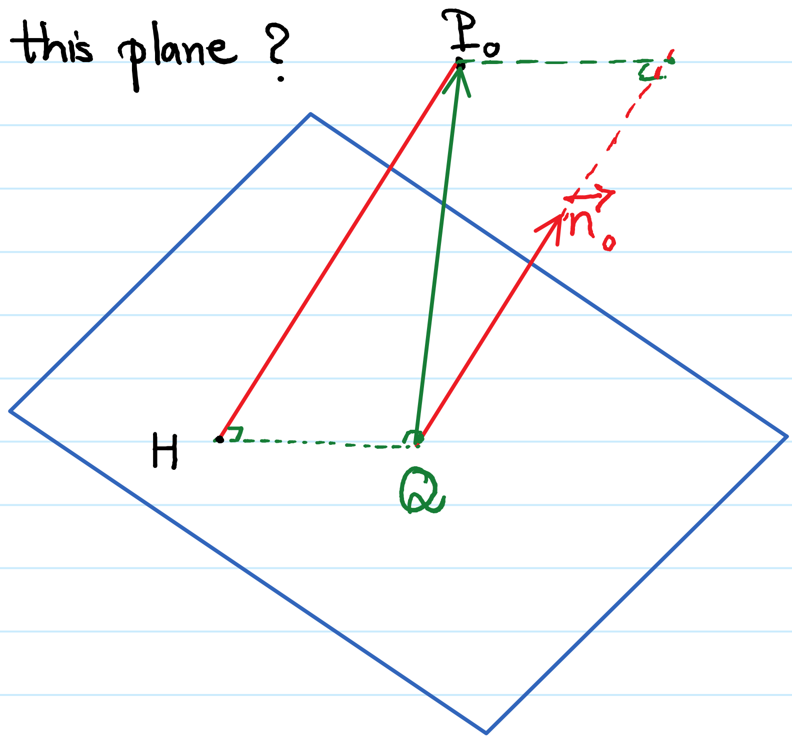
Solution. $(1,-1,1)$ is a common normal vector. So

$x - y + z = 1 - 1 + 1 = 1$ is an equation of this plane.

Lecture 8: Distance of a point from a plane

Monday, October 10, 2016 1:10 AM

<u>Question</u>. Given a point $P_0=(x_0,y_0,z_0)$ and a plane $ax+by+cz=d$. How can we find the distance of P_0 from this plane ?



Distance of P_0 from this plane $= P_0H$

$\overrightarrow{HP_0}$ is the orthogonal projection of $\overrightarrow{QP_0}$ along $\vec{n}_0$ where $Q=(x_1,y_1,z_1)$ is <u>any</u> point on this plane. Hence

$$\text{distance} = \left\| \operatorname{Proj}_{\vec{n}_0} \overrightarrow{QP_0} \right\| = \left| \frac{\vec{n}_0 \cdot \overrightarrow{QP_0}}{\vec{n}_0 \cdot \vec{n}_0} \right| \|\vec{n}_0\|$$

$$= \frac{\left| \vec{n}_0 \cdot \overrightarrow{OP_0} - \vec{n}_0 \cdot \overrightarrow{OQ} \right|}{\|\vec{n}_0\|^2} \cdot \|\vec{n}_0\|$$

Since $\vec{n}_0 \cdot \overrightarrow{OQ} = ax_1+by_1+cz_1 = d$, we have

Lecture 8: Distance of a point from a plane

Monday, October 10, 2016 1:25 AM

Overall we get

$$\boxed{\text{The distance of } P_0 = (x_0, y_0, z_0) \text{ from the plane } ax+by+cz=d \text{ is } \frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}}$$