Lecture 7: Geometric properties of cross product

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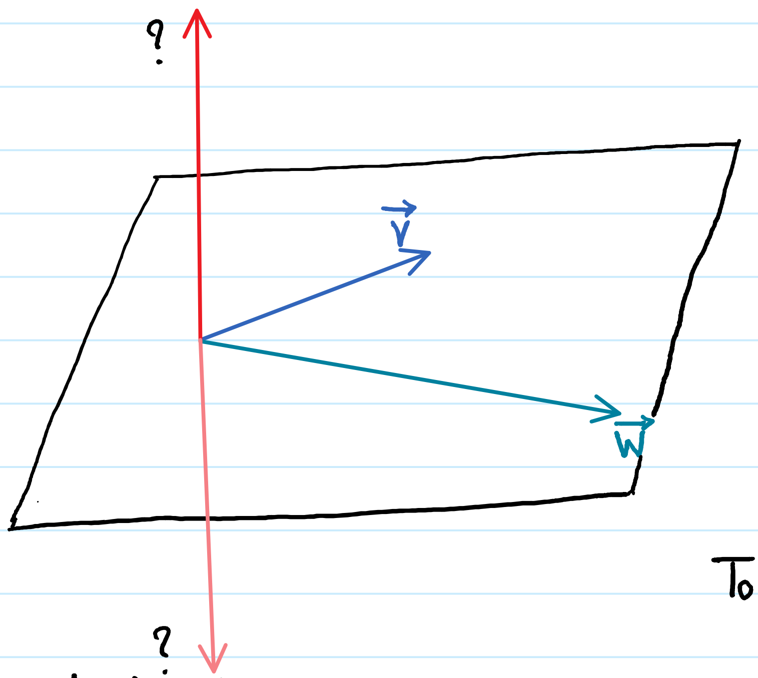
In the previous lecture we showed

$$(\vec{v}\times\vec{w})\cdot\vec{u} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{if} \quad \begin{aligned} \vec{v} &= (a_1,b_1,c_1), \\ \vec{w} &= (a_2,b_2,c_2), \\ \vec{u} &= (a_3,b_3,c_3). \end{aligned}$$

In particular, $(\vec{v}\times\vec{w})\cdot\vec{v} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = 0$

and $(\vec{v}\times\vec{w})\cdot\vec{w} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

So $\vec{v}\times\vec{w}$ is perpendicular to both $\vec{v}$ and $\vec{w}$. Therefore it is perpendicular to the plane spanned by $\vec{v}$ and $\vec{w}$.



Therefore $\vec{v}\times\vec{w}$ has two possible directions.

To determine which one is the correct direction, one has to use right-hand rule.

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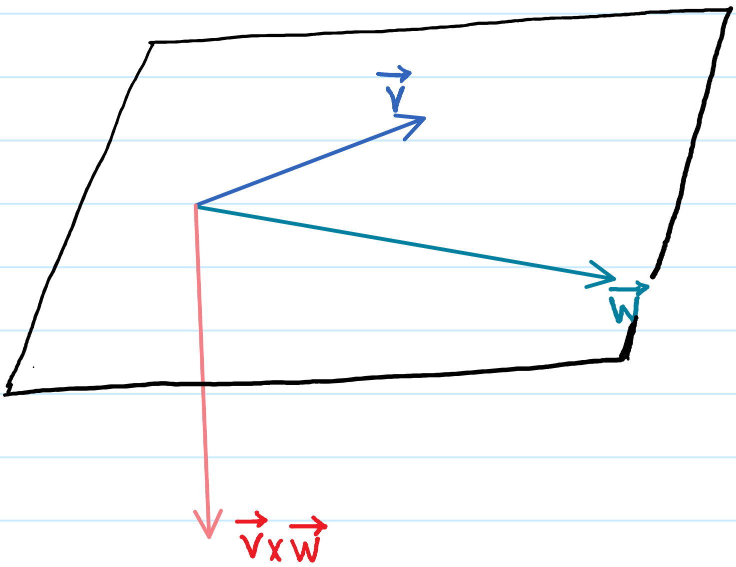
Fingers of your right hand should curl from the direction of $\vec{v}$ towards the direction of $\vec{w}$. Then your thumb gives you the direction of $\vec{v} \times \vec{w}$.



Now that we know how to find the direction of $\vec{v} \times \vec{w}$, let's look for its length. One can check the following equality:

$$\|\vec{v} \times \vec{w}\|^2 + |\vec{v} \cdot \vec{w}|^2 = \|\vec{v}\|^2 \|\vec{w}\|^2 .$$

$$\vec{v} \times \vec{w} = (b_1c_2 - b_2c_1,\ c_1a_2 - a_1c_2,\ a_1b_2 - b_1a_2)$$

$$\Rightarrow \|\vec{v} \times \vec{w}\|^2 + |\vec{v} \cdot \vec{w}|^2 = (b_1c_2 - b_2c_1)^2 + (c_1a_2 - a_1c_2)^2 + (a_1b_2 - b_1a_2)^2 + (a_1a_2 + b_1b_2 + c_1c_2)^2$$

$$= b_1^2c_2^2 + b_2^2c_1^2 \cancel{- 2b_1b_2c_1c_2} + c_1^2a_2^2 + a_1^2c_2^2 \cancel{- 2a_1a_2c_1c_2} + a_1^2b_2^2 + b_1^2a_2^2 \cancel{- 2a_1a_2b_1b_2} + a_1^2a_2^2 + b_1^2b_2^2 + c_1^2c_2^2 + \cancel{2a_1a_2b_1b_2} + \cancel{2a_1a_2c_1c_2} + \cancel{2b_1b_2c_1c_2} = (a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)$$

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[I have included the reasoning above, but in this you are expected to know only how to use the following conclusion.]

Hence $\|\vec{v}\times\vec{w}\|^2 = \|\vec{v}\|^2\|\vec{w}\|^2 - (\vec{v}\cdot\vec{w})^2$

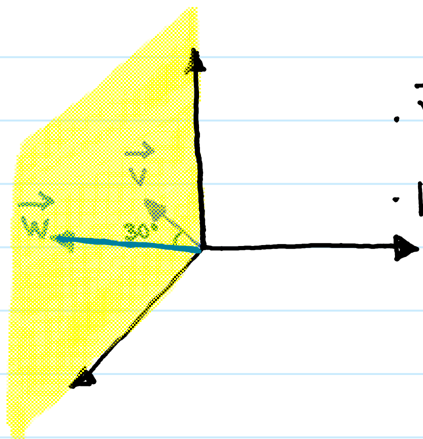
$$= \|\vec{v}\|^2\|\vec{w}\|^2 - \|\vec{v}\|^2\|\vec{w}\|^2 \cos^2\theta$$

$$= \|\vec{v}\|^2\|\vec{w}\|^2 (1-\cos^2\theta)$$

$$= \|\vec{v}\|^2\|\vec{w}\|^2 \sin^2\theta$$

Hence $\boxed{\|\vec{v}\times\vec{w}\| = \|\vec{v}\|\,\|\vec{w}\|\,\sin\theta}$
where θ is the angle between $\vec{v}$ and $\vec{w}$
(and $0\le\theta\le\pi$.)

Ex. Sketch $\vec{v}\times\vec{w}$ in the following figure:



- $\vec{v}$ and $\vec{w}$ are in xz-plane
- $\|\vec{v}\|=1$ and $\|\vec{w}\|=2$.

Solution. $\vec{v}\times\vec{w}$ is perpendicular to the plane spanned by $\vec{v}$ and $\vec{w}$. So it is perpendicular to the xz-plane. Hence it is parallel to the y-axis. Using the Right-Hand Rule, we see

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that $\vec{v} \times \vec{w}$ is towards the positive direction of y-axis.

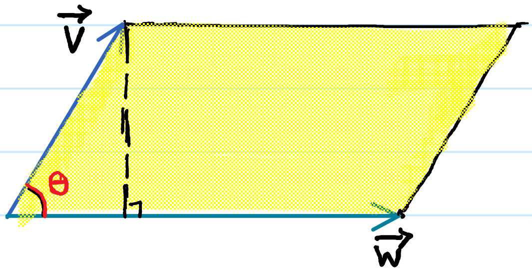
We also know

$$\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \, \|\vec{w}\| \sin\theta = (1)(2) \sin(30^\circ)$$

$$= (1)(2)(1/2) = 1.$$

Hence $\vec{v} \times \vec{w} = \vec{j}$.

Area of a parallelogram



$$\text{Area} = (\text{height})(\text{base})$$

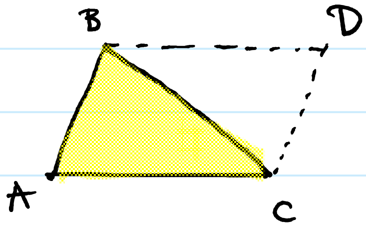
$$= (\|\vec{v}\| \sin\theta) \, \|\vec{w}\|$$

$$= \|\vec{v} \times \vec{w}\|.$$

Ex. Find the area of the triangle ABC where

$A = (1,2,0)$, $B = (0,1,2)$, $C = (1,1,1)$.

Solution. Area of ABC



$$= \frac{1}{2} \text{ area of } ABCD$$

$$= \frac{1}{2} \|\vec{AB} \times \vec{AC}\|$$

$\vec{AB} = (0,1,2) - (1,2,0) = (-1,-1,2)$.

$\vec{AC} = (1,1,1) - (1,2,0) = (0,-1,1)$.

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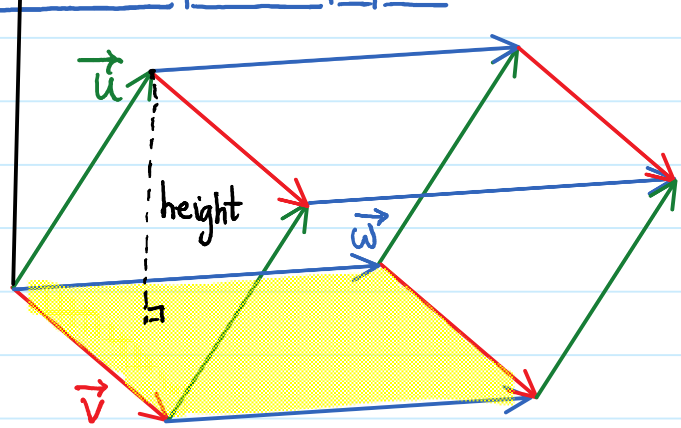
$$\vec{AB}\times\vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -1 & 2 \\ 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ -1 & 1 \end{vmatrix}\vec{i} - \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix}\vec{j} + \begin{vmatrix} -1 & -1 \\ 0 & -1 \end{vmatrix}\vec{k}$$

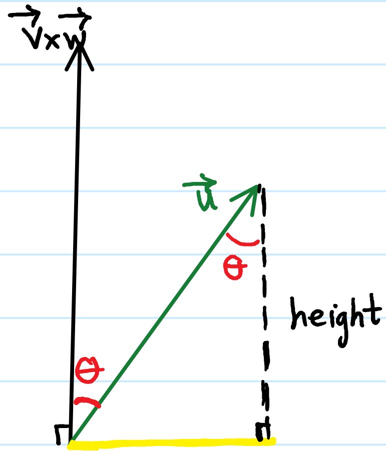
$$= (-1+2)\,\vec{i} + \vec{j} + \vec{k} = (1,1,1).$$

So $\|\vec{AB}\times\vec{AC}\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$.

Hence area of $ABC = \dfrac{\sqrt{3}}{2}$.

Volume of a parallelpiped.





volume $=$ (height) $\underbrace{(\text{area of the base})}$

area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$.

$=$ (height) $\|\vec{v}\times\vec{w}\|$.

$\vec{v}\times\vec{w}$ is perpendicular to the base. Hence $\vec{v}\times\vec{w}$ is parallel to the height. I am going to draw $\vec{v}\times\vec{w}$, $\vec{u}$, and the height separately (above figure on the right). So height $= \|\vec{u}\|\,|\cos\theta|$

where θ is the angle between $\vec{v}\times\vec{w}$ and $\vec{u}$. Thus

$$\text{volume} = (\|\vec{u}\|\,|\cos\theta|)\,\|\vec{v}\times\vec{w}\|$$
$$= \big|\,\|\vec{v}\times\vec{w}\|\,\|\vec{u}\|\cos\theta\,\big|$$
$$= |(\vec{v}\times\vec{w})\cdot\vec{u}|.$$

So we end up with the following formulas:

$$\boxed{\begin{array}{l} \text{volume} = |(\vec{v}\times\vec{w})\cdot\vec{u}| \\ \text{volume} = \left|\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\right| \\ \text{where } \vec{v}=(a_1,b_1,c_1),\ \vec{w}=(a_2,b_2,c_2) \\ \text{and } \vec{u}=(a_3,b_3,c_3). \end{array}}$$