Lecture 5: Geometric applications of dot product

Monday, October 3, 2016 8:36 AM

In the previous lecture we saw that

$\vec{v}\cdot\vec{w} = \|\vec{v}\|\,\|\vec{w}\|\cos\theta$ where θ is the angle between $\vec{v}$ and $\vec{w}$.

Ex. Find the angle between $\vec{v} = (1, \sqrt{2}, 1)$ and $\vec{w} = (1,0,0)$.

Solution. $\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}$. So we need to compute

$\vec{v}\cdot\vec{w}$, $\|\vec{v}\|$, and $\|\vec{w}\|$.

$\vec{v}\cdot\vec{w} = 1$, $\|\vec{v}\| = \sqrt{(1)^2 + (\sqrt{2})^2 + (1)^2} = 2$, $\|\vec{w}\| = 1$

So $\cos\theta = \dfrac{1}{2}$, which implies $\theta = \dfrac{\pi}{3}$.

Ex. Determine whether $\vec{v}$ and $\vec{w}$ are perpendicular or the angle between them is obtuse.

① $\vec{v} = (1,2,3)$ and $\vec{w} = (-1,-1,1)$.

② $\vec{v} = (1,2,3)$ and $\vec{w} = (1,-1,0)$.

③ $\vec{v} = (1,2,3)$ and $\vec{w} = (-1,0,1)$.

Solution. ① $\vec{v}\cdot\vec{w} = 0 \Rightarrow \vec{v}\perp\vec{w}$.

② $\vec{v}\cdot\vec{w} < 0 \Rightarrow \theta$ is obtuse.

③ $\vec{v}\cdot\vec{w} > 0 \Rightarrow \theta$ is acute.

Lecture 5: Orthogonal projection

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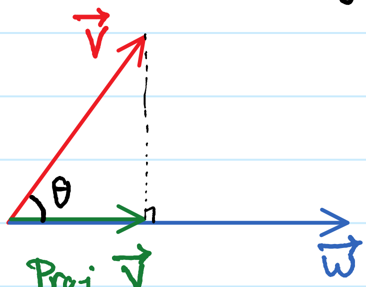
Let's start with a geometric interpretation of inner product.



$\text{Proj}_{\vec{w}} \vec{v}$ → (Orthogonal) Projection of $\vec{v}$ along $\vec{w}$.

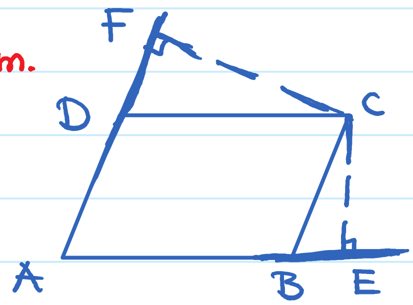
$$\|\text{Proj}_{\vec{w}} \vec{v}\| = \|\vec{v}\| \, |\cos\theta| = \|\vec{v}\| \frac{|\vec{v}\cdot\vec{w}|}{\|\vec{v}\| \, \|\vec{w}\|}$$

So $\|\text{Proj}_{\vec{w}} \vec{v}\| = \dfrac{|\vec{v}\cdot\vec{w}|}{\|\vec{w}\|}$ and $\boxed{|\vec{v}\cdot\vec{\omega}| = \|\vec{\omega}\| \, \|\text{Proj}_{\vec{\omega}} \vec{v}\|}$

And $\text{Proj}_{\vec{\omega}} \vec{v}$ is parallel to $\vec{\omega}$. So the unit vector in its direction is $\pm \dfrac{\vec{\omega}}{\|\vec{\omega}\|}$. Its sign depends on whether θ is acute or obtuse. Hence a closer examination implies $\text{Proj}_{\vec{\omega}} \vec{v} = \dfrac{\vec{v}\cdot\vec{\omega}}{\|\vec{\omega}\|} \, \dfrac{\vec{\omega}}{\|\vec{\omega}\|}$. And so

$$\boxed{\text{Proj}_{\vec{\omega}} \vec{v} = \frac{\vec{v}\cdot\vec{\omega}}{\vec{\omega}\cdot\vec{\omega}} \, \vec{\omega}}$$

Challenge Problem.



Show that

$AC^2 = (AB)(AE) + (AD)(AF).$

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Ex. Find all values of x for which

$$(x,1,2) \perp (-1,2,x).$$

Solution. $(x,1,2) \perp (-1,2,x)$ if and only if

$$0 = (x,1,2)\cdot(-1,2,x) = -x + 2 + 2x$$

And so $x = 2$.

Ex. Find the projection of $\vec{v} = (1,2,3)$ along $\vec{w} = (2,0,-1)$.

Solution. $\text{Proj}_{\vec{w}} \vec{v} = \dfrac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\,\vec{w}$

$$= \frac{(1)(2)+(2)(0)+(3)(-1)}{(2)(2)+(0)(0)+(-1)(-1)}\,(2,0,-1)$$

$$= \left(\frac{-2}{5},\, 0,\, \frac{1}{5}\right).$$