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- For $SL_2$, we showed that $\frac{1}{x}\, dx \wedge dy \wedge dz$ is an $SL_2$ invariant gauge form where $\begin{bmatrix} t & z \\ y & x \end{bmatrix} \in SL_2$.

Now we find an $SL_n$-invariant guage form $\omega$, and at the same time compare the induced Haar measure on $SL_n(\mathbb{R})$ with the one given by Siegel w.r.t. which we have $\mu\left(SL_n(\mathbb{R})/SL_n(\mathbb{Z})\right) = \frac{1}{n} \prod_{k=2}^{n} \zeta(k)$.

- By definition we have

$$\int_{SL_n(\mathbb{R})} f(g)\, d\mu(g) = \int_{C_{SL_n(\mathbb{R})}} f(\det(Y)^{-1/n} Y)\, dY.$$

we make the following change of variables:

For $(i,j) \neq (n,n)$, let $y_{ij} = r^{1/n} x_{ij}$, and let $r = \det(Y)$. So

$$\bigwedge_{(i,j)\neq(n,n)} dy_{ij} \wedge dr = \det(Y_{nn})\, dy_{11} \wedge \cdots \wedge dy_{nn}$$

where $Y_{nn} = \begin{bmatrix} y_{11} & \cdots & y_{1(n-1)} \\ \vdots & \ddots & \vdots \\ y_{(n-1)1} & \cdots & y_{(n-1)(n-1)} \end{bmatrix}$.

$$dy_{ij} = (* dr + r^{1/n} dx_{ij}) \Rightarrow$$

$$\bigwedge_{(i,j)\neq(n,n)} dy_{ij} \wedge dr = r^{\frac{n^2-1}{n}} \bigwedge_{(i,j)\neq(n,n)} dx_{ij} \wedge dr$$

$$= \det(Y_{nn})\, dy_{11} \wedge \cdots \wedge dy_{nn}$$

$$= \det(Y_{nn})\, dy_{11}\wedge\cdots\wedge dy_{nn}$$

$$= r^{\frac{n-1}{n}} \det(X_{nn})\, dy_{11}\wedge\cdots\wedge dy_{nn}$$

$$\Rightarrow dY = r^{\frac{n^2-n}{n}} \cdot \det(X_{nn})^{-1} \cdot \bigwedge_{(i,j)\neq(n,n)} dx_{ij} \wedge dr$$

$$\Rightarrow \int_{SL_n(\mathbb{R})} f(g)\, d\mu(g) = \int_{SL_n(\mathbb{R})} \int_0^1 f(x) \cdot r^{n-1} \cdot |\det(X_{nn})|^{-1}\, dr\, dX^*$$

$$= \int_{SL_n(\mathbb{R})} \underbrace{\left(\int_0^1 r^{n-1}\, dr\right)}_{1/n} f(x)\, |\det(X_{nn})|^{-1}\, dx_{11} \cdots dx_{n,n-1}$$

$$\Rightarrow \mu = \frac{1}{n} |\det(X_{nn})|^{-1}\, dx_{11} \cdots dx_{n,n-1}$$

$$\Rightarrow \omega = \det(X_{nn})^{-1}\, dx_{11}\wedge\cdots\wedge dx_{n,n-1}$$ is $SL_n$-invariant gauge form,

where $X_{nn} = \begin{bmatrix} x_{11} & \cdots & x_{1n-1} \\ \vdots & \ddots & \vdots \\ x_{n-1,1} & \cdots & x_{n-1,n-1} \end{bmatrix}$.

Let $|\omega|_\infty$ be the Haar measure induced by $\omega$. So

$$|\omega|_\infty \left( SL_n(\mathbb{R}) / SL_n(\mathbb{Z}) \right) = \zeta(2) \cdot \zeta(3) \cdot \ldots \cdot \zeta(n).$$

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p-adic numbers,

In order to define and compute Tamagawa number of $(SL_n)_{\mathbb{Q}}$, we need to define p-adic and adelic numbers, and prove special case of strong approximation.

special case of strong approximation.

For any prime $p$, let $v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ be the p-adic valuation, i.e. $\forall\, a \in \mathbb{Q} \setminus \{0\}$, $a\mathbb{Z} = \prod (p\mathbb{Z})^{v_p(a)}$

and $v_p(0) = \infty$.

Let $|a|_p := \left(\frac{1}{p}\right)^{v_p(a)}$. So
$$\prod_{p \in \mathcal{P} \cup \{\infty\}} |a|_p = |a|_\infty \cdot \prod_{p \in \mathcal{P}} \left(\frac{1}{p}\right)^{v_p(a)}$$
$$= \prod p^{v_p(a)} \cdot \prod \left(\frac{1}{p}\right)^{v_p(a)}$$
$$= 1.$$

(special case of product formula.)

Basic properties of the p-adic norm.

- $|a|_p = 0 \iff a = 0$, and $|a|_p \geq 0$.
- $|a+b|_p \leq \max\{|a|_p, |b|_p\} \leq |a|_p + |b|_p$.
- $|ab|_p = |a|_p \cdot |b|_p$.

Now we complete $\mathbb{Q}$ with respect to this norm. I.e.

- $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence if $\forall\, \varepsilon > 0$, $m,n \gg_\varepsilon 1$,
$$|a_n - a_m|_p \leq \varepsilon.$$
- $\{\alpha_n\}_{n=1}^{\infty}$ is a Null sequence if $\forall\, \varepsilon > 0$, $n \gg_\varepsilon 1$,
$$|a_n|_p \leq \varepsilon.$$

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- Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two Cauchy sequences.
  We say $\{a_n\}_{n=1}^{\infty} \sim \{b_n\}_{n=1}^{\infty}$ if $\{a_n - b_n\}_{n=1}^{\infty}$ is a Null sequence.
- $\mathbb{Q}_p := \{ [\{a_n\}_{n=1}^{\infty}]_{\sim} \mid \{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence $\}$.

<u>Basic properties</u>

- $v_p$ can be extended to $\mathbb{Q}_p$ ;

  $v_p : \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}$ with the same property as above.

$\Rightarrow$ - $\mathbb{Z}_p := \{ x \in \mathbb{Q}_p \mid v_p(x) \geq 0 \}$ is a subring.

- $p\mathbb{Z}_p = \{ x \in \mathbb{Q}_p \mid v_p(x) > 0 \}$ is the unique maximal ideal of $\mathbb{Z}_p$

- $\mathbb{Z}_p^{\times} = \{ x \in \mathbb{Q}_p \mid v_p(x) = 0 \}$

- $\alpha \in \mathbb{Q}_p^{\times} \Rightarrow \alpha \in p^{v_p(\alpha)} \mathbb{Z}_p^{\times}$

- $p\mathbb{Z}_p \underset{\log}{\overset{\exp}{\rightleftarrows}} 1 + p\mathbb{Z}_p$ $\qquad$ $p$ : odd prime

  $4\mathbb{Z}_2 \underset{\log}{\rightleftarrows} 1 + 4\mathbb{Z}_2$

  are well-defined analytic maps.

- $$1 \to 1 + p\mathbb{Z}_p \to \mathbb{Z}_p^{\times} \to \mathbb{F}_p^{\times} \to 1$$

  splits by Hensel's lemma.

• Hensel's lemma  Let $f_1, \cdots, f_m \in \mathbb{Z}_p[t_1, \cdots, t_n]$.

Suppose $$[\partial_j f_i(x_0)]\, \mathbb{Z}_p^n = \mathbb{Z}_p^m$$

and $$f_1(x_0) \equiv f_2(x_0) \equiv \cdots \equiv f_m(x_0) \overset{p}{\equiv} 0$$

where $p$ is an odd prime.

$$\Rightarrow \exists x \in \mathbb{Z}_p^n \text{ s.t. } x \overset{p}{\equiv} x_0$$

$$\text{and } f_1(x) = \cdots = f_m(x) = 0.$$

Remark. This says if $f_i$'s are small at $x_0$, then $x_0$ is actually close to a common root. The method of proof is similar to Newton's method.

Proof. We construct $\{x_k\}_{k=1}^{\infty}$ inductively such that

① $f_i(x_k) \overset{p^{k+1}}{\equiv} 0$.

② $x_k \overset{p^k}{\equiv} x_{k-1}$.

Having such $\{x_k\}$, by ② it is a Cauchy sequence.

Let $\bar{x}$ be its limit. Then $f_i(\bar{x}) = 0$, $\forall i$. And

$$\text{For } k \gg 0, \quad \left.\begin{array}{l} \bar{x} \overset{p}{\equiv} x_k \\ x_k \overset{p}{\equiv} x_0 \end{array}\right\} \Rightarrow \bar{x} \overset{p}{\equiv} x_0.$$

$$\cdot\ f_i(x_{k-1} + p^k t) = f_i(x_{k-1}) + p^k \nabla f_i(x_{k-1}) \cdot t$$

$$+ \sum_I \frac{\partial_I f_i(x_{k-1})}{I!} p^{k\|I\|_1} t^I$$

$p$

$$v_p\left(\frac{\partial^I f(x_{k-1})}{I!}\, p^{k|I|}\right) \geq k|I| - v_p(|I|!) \geq k+1$$

$$v_p(s!) = \sum_{j=1}^{\infty} \left\lfloor \frac{s}{p^j} \right\rfloor \leq \sum_{j=1}^{\infty} \frac{s}{p^j} = \frac{s}{p} \Big/ (1 - 1/p)$$

$$ks - v_p(s!) > ks - \frac{s}{p-1} \underset{k \geq 1,\ s \geq 2}{\geq} 2k - \frac{2}{p-1} \underset{p \geq 3,\ k \geq 1}{\geq} k$$

$$\left[\nabla_i f(x_{k-1})\right] t = \left[f_i(x_{k-1})/p^k\right] \in \mathbb{Z}_p^n \quad \text{has}$$

a solution by our assumption (why ?). So we are done. ∎

- $\mathbb{F}_p^\times$ is cyclic $\rightsquigarrow \exists\, x_0 \in \mathbb{Z}_p$ s.t. $x_0^{p-1} - 1 \overset{p}{\equiv} 0$

  and its derivative is $(p-1)\, x_0^{p-2} \in \mathbb{Z}_p^\times$

  $\rightsquigarrow$ By Hensel, $\exists\, x \in \mathbb{Z}_p$ s.t. $x^{p-1} = 1$

  and $x \overset{p}{\equiv} x_0$.

- So $\mathbb{Z}_p^\times \simeq \mathbb{F}_p^\times \times \mathbb{Z}_p$ if $p$ is odd.

- Corollary. $V/\mathbb{Z}_p$ is a smooth affine $\mathbb{Z}_p$-scheme

  $\Rightarrow V(\mathbb{Z}_p) \longrightarrow V(\mathbb{F}_p)$ is onto.

- Corollary $SL_n(\mathbb{Z}_p) \longrightarrow SL_n(\mathbb{F}_p)$ is onto.

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$\boxed{Q}$ What is $|\omega|_p(SL_n(\mathbb{Z}_p))$ ?

$\boxed{A}$ As in the $SL_2$ case, let $SL^1(\mathbb{Z}_p) := \ker(SL_n(\mathbb{Z}_p) \to SL(\mathbb{F}_p))$.

$\boxed{A}$ As in the $SL_2$ case, let $SL_n^1(\mathbb{Z}_p) := \ker(SL_n(\mathbb{Z}_p) \to SL_n(\mathbb{F}_p))$.

By the above Corollary, $[SL_n(\mathbb{Z}_p) : SL_n^1(\mathbb{Z}_p)] = |SL_n(\mathbb{F}_p)|$. So

$$|\omega|_p(SL_n(\mathbb{Z}_p)) = |SL_n(\mathbb{F}_p)| \cdot |\omega|_p(SL_n^1(\mathbb{Z}_p))$$

We will prove that for any $(t_{ij})_{(i,j)\neq(n,n)} \in \mathbb{Z}_p$, there is $t_{nn} \in \mathbb{Z}_p$ s.t. $1 + pT \in SL_n(\mathbb{Z}_p)$ where $T = [t_{ij}]$.

For any $i$, let $\hat{T}_{in}$ be the $(n-1)\times(n-1)$ matrix after removing the $i^{th}$ row and the $n^{th}$ column from $1+pT$. Notice that

$$\hat{T}_{in} \overset{p}{\equiv} \begin{array}{c} \left[\begin{array}{ccc|cccc} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ \hline & & & 0 & 1 & & \\ & & & & \ddots & \ddots & \\ & & & & & & 1 \\ & & & & & & 0 \end{array}\right] \begin{array}{l} \}\, i-1 \\ \\ \\ \}\, n-i \\ \\ \\ \\ \end{array} \\ \underbrace{\qquad\quad}_{i-1}\underbrace{\qquad\quad\quad}_{n-i} \end{array}$$

$$\Rightarrow \det \hat{T}_{in} \overset{p}{\equiv} \begin{cases} 0 & \text{if } i<n \\ 1 & \text{if } i=n \end{cases}$$

On the other hand,
$$\det(1+pT) = \sum_{i=1}^{n} (-1)^{n-i} (\delta_{in} + pt_{in}) \det(\hat{T}_{in})$$
$$= (1+pt_{nn}) \det(\hat{T}_{nn}) + p \sum_{i=1}^{n-1} (-1)^{n-i} t_{in} \det(\hat{T}_{in})$$

$$\Rightarrow 1+pt_{nn} = \det(\hat{T}_{nn})^{-1} \left(1 - p \sum_{i=1}^{n-1} (-1)^{n-i} t_{in} \det(\hat{T}_{in})\right)$$

$$\Rightarrow 1+pt_{nn} = \underbrace{\det(T_{nn})}_{\in \; 1+p\mathbb{Z}_p} \quad \underbrace{\left(1 - p \sum_{i=1} (-1) \; t_{in} \det(T_{in})\right)}_{\in \; 1+p\mathbb{Z}_p}$$

$\Rightarrow$ it has a solution in $\mathbb{Z}_p$.

$\Rightarrow (t_{ij})_{(i,j)\neq(n,n)} \in \mathbb{Z}_p$ give us a coordinate system for $SL_n^1(\mathbb{Z}_p)$

$$\Rightarrow |\omega|_p(SL_n^1(\mathbb{Z}_p)) = \int_{\mathbb{Z}_p^{n^2-1}} |\det(1+pT_{nn})|_p^{-1} \, |p^{n^2-1}|_p \, dt_{11} \cdots dt_{nn-1}$$

$$= \frac{1}{p^{n^2-1}}.$$

$$\Rightarrow |\omega|_p(SL_n(\mathbb{Z}_p)) = \frac{|SL_n(\mathbb{F}_p)|}{p^{n^2-1}} = \frac{\prod_{i=1}^{n} (p^n - p^i)}{p^{n^2-1}(p-1)} = \prod_{i=2}^{n} (1-p^{-i}).$$

∎