We computed the covolume of $SL_2(\mathbb{Z})$ in $SL_2(\mathbb{R})$ w.r.t. several volume forms. Our main tool, however, was having a precise fundamental domain in the symmetric space $\mathcal{H}$ of $SL_2(\mathbb{R})$.

- Is $SL_n(\mathbb{Z})$ a lattice in $SL_n(\mathbb{R})$?
- Can we compute its covolume w.r.t. a "natural" volume form?

This is called <u>Minkowski's reduction theory</u>. He was studying positive definite quadratic forms, and he was interested in <u>$Q(\mathbb{Z}^n)$</u>. So he was allowed to change $Q(v)$ to $Q(\gamma v)$ where $\gamma \in GL_n(\mathbb{Z})$. Following Gauss, he described "a reduced form of a quadratic form".

We will be following modern treatment of this subject by Siegel.

<u>Def</u>. Let $\Omega(\mathbb{R}^n) := \{\Delta \subseteq \mathbb{R}^n \mid \Delta \text{ is a lattice in } \mathbb{R}^n\}$,

$$\Omega^{(1)}(\mathbb{R}^n) := \{\Delta \in \Omega(\mathbb{R}^n) \mid \ell(\mathbb{R}^n/\Delta) = 1 \text{, where } \ell \text{ is the Lebesgue measure}\}.$$

<u>Modified Reduction Theory</u>.

<u>Lemma</u>. Suppose $W \subsetneq V$ is a proper subspace.

If $\Delta \in \Omega(V)$, $\Delta \cap W \supseteq \bigoplus_{i=1}^{k} \mathbb{Z} v_i \in \Omega(W)$, and $W = \bigoplus_{i=1}^{k} \mathbb{R} v_i$, then $\Pr_{W^\perp}(\Delta)$ is a discrete subgroup

<u>Pf.</u> • By our assumption $\mathcal{F} := \{\sum_{i=1}^{k} c_i v_i \mid |c_i| \leq 1/2\}$

Pf. • By our assumption $\mathcal{F} := \{ \sum_{i=1} c_i v_i \mid |c_i| \leq 1/2 \}$ contains a fundamental domain of $\Delta \cap W$ in W, and we notice that $\mathcal{F}$ is compact.

• Suppose that for some $x_i \in \Delta$, $Pr_{W^\perp}(x_i) \xrightarrow[i\to\infty]{} 0$, and $Pr_{W^\perp}(x_i) \neq Pr_{W^\perp}(x_j)$.

$Pr_W(x_i) = \delta_i + \sigma_i$ where $\delta_i \in \Delta \cap W$ and $\sigma_i \in \mathcal{F}$.

Since $\mathcal{F}$ is compact, passing to a subseq. we can and will assume that $\sigma_i \xrightarrow[i\to\infty]{} \sigma \in \mathcal{F}$.

So we have
$$x_i - \delta_i = Pr_W(x_i - \delta_i) + Pr_{W^\perp}(x_i - \delta_i)$$
$$= \sigma_i + Pr_{W^\perp}(x_i) \xrightarrow[i\to\infty]{} \sigma \in \mathcal{F}$$

On the other hand $x_i - \delta_i \in \Delta$ and Δ is discrete $\Rightarrow$ for large enough $\underline{i}$, $x_i - \delta_i = \sigma \Rightarrow Pr_{W^\perp}(x_i - \delta_i) = 0$ $\Rightarrow Pr_{W^\perp}(x_i) = 0$, which is a contradiction. ■

• In put $\Delta \in \Omega(\mathbb{R}^n)$.

• Out put $v_1, \cdots, v_n \in \Delta$

• Auxiliary variables: $\Delta_i = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_i$ and
$$V_i = \mathbb{R}v_1 \oplus \cdots \oplus \mathbb{R}v_i .$$
$$\Delta_0 = 0 \text{ and } V_0 = 0 .$$

- For $i=0$ to $n-1$

  1 $*$ $v_{i+1} \in \Delta$ s.t. $\|Pr_{V_i^\perp}(v_{i+1})\| = \min\{\|x\| \mid x \in Pr_{V_i^\perp}(\Delta) \setminus \{0\}\}$

  and among all such v's in Δ choose a vector s.t. $\|v\|$ is smallest possible.

  $*$ $\Delta_{i+1} := \Delta_i \oplus \mathbb{Z} v_{i+1}$

  $*$ $V_{i+1} := V_i \oplus \mathbb{R} v_{i+1}$

1 is doable because by Lemma $Pr_{V_i^\perp}(\Delta)$ is discrete.

<u>Proposition</u> For $1 \le i \le n$, $\Delta \cap V_i = \Delta_i$.

<u>Pf.</u> We use induction on i. The base case is clear.

<u>Inductive step.</u> $v \in \Delta \cap V_{k+1} \Rightarrow$ W.L.O.G. we can and will assume $v = \sum_{i=1}^{k+1} c_i v_i$ s.t. $|c_i| \le 1/2$.

By the definition of v_{k+1}, $\|Pr_{V_k^\perp}(v_{k+1})\| \le \|Pr_{V_k^\perp}(v)\|$ <u>or</u>

$Pr_{V_k^\perp}(v) = 0$. $\circledast$

$$Pr_{V_k^\perp}(v) = \sum_{i=1}^{k+1} c_i \, Pr_{V_k^\perp}(v_i) = c_{k+1} \, Pr_{V_k^\perp}(v_{k+1})$$

$$\Rightarrow \|Pr_{V_k^\perp}(v)\| = |c_{k+1}| \, \|Pr_{V_k^\perp}(v_{k+1})\| \le \frac{1}{2} \|Pr_{V_k^\perp}(v_{k+1})\|$$

$$\|\Pr_{V_k^\perp}(v)\| \quad \| \quad v_{k+1}\| \quad \|\Pr_{V_k^\perp}(v_{k+1})\| - 2\,\|\Pr_{V_k^\perp}(v_{k+1})\|$$

$$\circledast \Longrightarrow \Pr_{V_k^\perp}(v) = 0 \Longrightarrow v \in V_k.$$

By induction hypothesis, $\Delta \cap V_k = \bigoplus_{i=1}^{k} \mathbb{Z} v_i$ $\Bigg\} \Rightarrow v \in \bigoplus_{i=1}^{k+1} \mathbb{Z} v_i = \Delta_{k+1}$ ■

<u>Corollary</u> $\Delta \in \Omega(\mathbb{R}^n) \Rightarrow \Delta = \bigoplus_{i=1}^{n} \mathbb{Z} v_i$ for some v_i's.

<u>Def.</u> <u>Chabauty topology on $\Omega(\mathbb{R}^n)$</u>. For Δ_m's, $\Delta \in \Omega(\mathbb{R}^n)$

We say $\Delta_m \xrightarrow[m\to\infty]{} \Delta$ if for any compact subset C of $\mathbb{R}^n$

$\Delta_m \cap C \longrightarrow \Delta \cap C$ in Hausdorff distance, i.e.

$$d_{\mathrm{Hau.}}(\Delta_m \cap C, \Delta \cap C) := \inf\{ r \in \mathbb{R}^+ \mid r\text{-nbhd of } \Delta_m \cap C \supseteq \Delta \cap C \ \underline{\text{and}}\ r\text{-nbhd of } \Delta \cap C \supseteq \Delta_m \cap C. \} \xrightarrow[m\to\infty]{} 0$$

<u>Lemma</u>. $$GL_n(\mathbb{R})/GL_n(\mathbb{Z}) \xrightarrow{\Theta} \Omega(\mathbb{R}^n)$$

$$\Theta(g\, GL_n(\mathbb{Z})) := g\mathbb{Z}^n$$

is a bijection.

<u>Pf.</u> • For $g \in GL_n(\mathbb{R})$, $g\mathbb{Z}^n$ is discrete and

$\mathrm{vol}\left(\mathbb{R}^n/g\mathbb{Z}^n\right) = |\det(g)| < \infty$. And $\{ g \in GL_n(\mathbb{R}) \mid g\mathbb{Z}^n = \mathbb{Z}^n \} = GL_n(\mathbb{Z})$ (?)

So Θ is well-defined and 1-1.

So Θ is well-defined and 1-1.

• By the above corollary, Θ is onto.

Clearly Θ is a $GL_n(\mathbb{R})$-equivariant map, i.e.

$$\forall g, g' \in GL_n(\mathbb{R}), \quad g\,\Theta(g'\Gamma) = \Theta(gg'\Gamma). \quad \blacksquare$$

[We also discussed that $\mathrm{Pr}_{\ell}(\mathbb{Z}^2)$ is dense in ℓ if $\ell = \mathbb{R}\begin{bmatrix}1\\ \alpha\end{bmatrix}$ and $\alpha \in \mathbb{R}\setminus\mathbb{Q}$; it is proved using the fact that $\{n\alpha\}_{n=1}^{\infty}$ is dense in $[0,1]$.]