Lecture6: fundamental domain and SL(2,Z)

Tuesday, October 6, 2015 12:50 AM

Let's go back to $\mathcal{H}$. We said that on an orientable surface Σ we can put a hyperbolic structure if and only if $\pi_1(\Sigma)$ has a discrete faithful rep'n into $PSL_2(\mathbb{R})$.

Def. A discrete subgroup of $SL_2(\mathbb{R})$ is called a Fuchsian group.

Def. $\mathcal{F} \subseteq \mathcal{H}$ is called a fundamental domain of Γ if

⓪ $\mathcal{F} = \overline{\mathcal{F}^\circ}$: closure of its interior.

① $\mathcal{H} = \bigcup_{\gamma \in \Gamma} \gamma \mathcal{F}$

② $\mathcal{F}^\circ \cap \gamma \mathcal{F} = \emptyset$ for $\gamma \neq id$.

③ $|\{\gamma \in \Gamma \mid \mathcal{F} \cap \gamma \mathcal{F} \neq \emptyset\}| < \infty$.

Lemma If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two fundamental domains of Γ and $vol_{\mathcal{H}}(\partial \mathcal{F}_i) = 0$, then $vol_{\mathcal{H}}(\mathcal{F}_1) = vol_{\mathcal{H}}(\mathcal{F}_2)$.

Pf. $\mathcal{F}_1^\circ = \bigcup_{\gamma \in \Gamma} (\mathcal{F}_1^\circ \cap \gamma \mathcal{F}_2) \Rightarrow vol_{\mathcal{H}}(\mathcal{F}_1^\circ) \leq \sum_{\gamma \in \Gamma} vol_{\mathcal{H}}(\mathcal{F}_1^\circ \cap \gamma \mathcal{F}_2)$

$$= \sum_{\gamma \in \Gamma} vol_{\mathcal{H}}(\gamma^{-1} \mathcal{F}_1^\circ \cap \mathcal{F}_2)$$

$$= vol_{\mathcal{H}}\left(\left(\bigcup_{\gamma \in \Gamma} \gamma \mathcal{F}_1^\circ\right) \cap \mathcal{F}_2\right)$$

$$\leq vol_{\mathcal{H}}(\mathcal{F}_2).$$

$\Rightarrow vol_{\mathcal{H}}(\mathcal{F}_1) \leq vol_{\mathcal{H}}(\mathcal{F}_2)$. ∎

Lemma. Γ : Fuchsian $\Rightarrow \{x \in \mathcal{H} \mid \mathrm{Stab}_{\Gamma} x \neq \mathrm{Id}\}$ is discrete.

Pf. If not, $\exists\, x_n \in \mathcal{H},\ \gamma_n \in \Gamma \setminus \{1\}$ s.t.

① $x_n \longrightarrow x$

② $\gamma_n \cdot x_n = x_n$

$\Rightarrow\ 0 \longleftarrow d_{\mathcal{H}}(x_n, x) = d_{\mathcal{H}}(\gamma_n \cdot x_n, \gamma_n \cdot x) = d_{\mathcal{H}}(x_n, \gamma_n \cdot x)$

$\Rightarrow\ d_{\mathcal{H}}(\gamma_n \cdot x, x) \leq 2\, d_{\mathcal{H}}(x, x_n) \longrightarrow 0.$

Since $\Gamma \cdot x$ is discrete, for $n \geq n_0$ $\gamma_n \cdot x = \gamma_{n_0} \cdot x$

Since Γ is discrete and $\mathrm{Stab}_G\, x$ is compact, $\gamma_n = \gamma_{n_0'}$ for $n \geq n_0'$.

Since $\gamma_{n_0'} \in SL_2(\mathbb{R})$ and it fixes infinitely many points, it is 1,

which is a contradiction. ■

Proposition (Dirichlet domain)

Suppose Γ is a Fuchsian group, $x_0 \in \mathcal{H}$ s.t. $\mathrm{Stab}_{\Gamma}(x_0) = \{1\}$.

Let $D(x_0; \Gamma) := \{x \in \mathcal{H} \mid \forall \gamma \in \Gamma,\ \gamma \neq 1,\ d_{\mathcal{H}}(\gamma \cdot x_0, x) \geq d_{\mathcal{H}}(x_0, x)\}$.

Then $D(x_0; \Gamma)$ is a fundamental domain of Γ.

Pf. Let $\mathcal{F} := D(x_0; \Gamma)$.

Step 0. $\mathcal{F}$ is intersection of half-planes (why?). So it is convex, closed, and path connected.

$\mathcal{F}^\circ = \{x \in \mathcal{H} \mid \forall \gamma \in \Gamma \setminus 1,\ d_{\mathcal{H}}(\gamma \cdot x_0, x) > d_{\mathcal{H}}(x_0, x)\}$

$(\subseteq)$ (easy)

$(\supseteq)$ $B\,(d_{\mathcal{H}}(\Gamma \cdot x, x)/2) \subset \mathcal{F}^\circ$

$(\subseteq)$ (easy)

$(\supseteq)$ $B_x(d_{\mathcal{H}}(\Gamma\cdot x_0, x)/2) \subseteq \mathcal{F}^\circ$

$\overline{\mathcal{F}^\circ} = \mathcal{F}$ (nbhd of a point intersects only finitely many of the above planes.)

<u>Step 1.</u> $\mathcal{H} = \bigcup_{\gamma\in\Gamma} \gamma \mathcal{F}$.

$\forall x \in \mathcal{H}$, since $\Gamma\cdot x_0$ is discrete, $\exists \gamma_0 \in \Gamma$ s.t.

$$d_{\mathcal{H}}(\gamma_0\cdot x_0, x) = d_{\mathcal{H}}(\Gamma\cdot x_0, x)$$

$$\Rightarrow \quad d_{\mathcal{H}}(x_0, \gamma_0^{-1}\cdot x) = d_{\mathcal{H}}(\Gamma\cdot x_0, \gamma_0^{-1}\cdot x)$$

$$\Rightarrow \quad \gamma_0^{-1}\cdot x \in \mathcal{F} \quad \Rightarrow \quad x \in \gamma_0 \mathcal{F}.$$

<u>Step 2.</u> $\gamma \mathcal{F}^\circ \cap \mathcal{F} = \emptyset$ if $\gamma \neq 1$.

Suppose $\exists x \in \mathcal{F}^\circ$ and $\gamma \in \Gamma \setminus 1$ s.t. $\gamma\cdot x \in \mathcal{F}$

$$\Rightarrow \quad \underset{x\in\mathcal{F}^\circ}{d(x_0, x) < d(\gamma^{-1}\cdot x_0, x)} = d(x_0, \gamma\cdot x)$$

and

$$\underset{\gamma\cdot x\in\mathcal{F}}{d(x_0, \gamma\cdot x) \leq d(\gamma\cdot x_0, \gamma\cdot x)} = d(x_0, x) \quad \Rightarrow \divideontimes$$

■

<u>Proposition</u> $\mathcal{F} := \{ z \in \mathcal{H} \mid |z| \geq 1, \ |\mathrm{Re}\, z| \leq 1/2 \}$ is a fundamental domain of $PSL_2(\mathbb{Z})$.

<u>Pf.</u> • $\mathrm{Stab}_G(\alpha i) \cap PSL_2(\mathbb{Z}) \ni \{\pm g\}$.

$\alpha > 1$

$$\frac{a(\alpha i)+b}{c(\alpha i)+d} = \alpha i \quad \Rightarrow \quad \alpha a i + b = -\alpha^2 c + \alpha d i$$

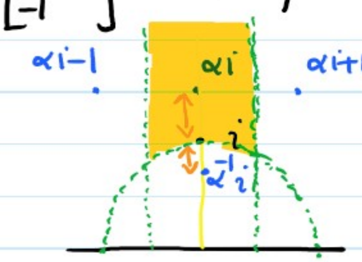
$$\Rightarrow \begin{cases} \alpha a = \alpha d \\ b = -\alpha^2 c \end{cases} \Rightarrow \begin{cases} a = d \\ b = -\alpha^2 c \end{cases}$$

$$\Rightarrow \quad 1 = ad - bc$$
$$= a^2 + \alpha^2 b^2$$

$$\Rightarrow \quad b = 0 \quad \text{and} \quad a = \pm 1.$$

$$\Rightarrow \mathrm{Stab}_{\Gamma}(\alpha i) = I.$$

• $$D(\alpha i;\Gamma) \ni z \Rightarrow \left|\begin{array}{l} d_{\mathcal{H}}(\alpha i, z) \le d_{\mathcal{H}}\left(\begin{bmatrix} 1 & \pm 1 \\ & 1 \end{bmatrix}(\alpha i), z\right) \\ \quad \Rightarrow |\mathrm{Im}(z)| \le 1/2. \\ d_{\mathcal{H}}(\alpha i, z) \le d_{\mathcal{H}}\left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}(\alpha i), z\right) \\ \quad \Rightarrow |z| \ge 1. \end{array}\right.$$



• In order to show the equality it is enough to show there are no $z \in \mathcal{F}^\circ$ and $\gamma \in \Gamma \setminus \{1\}$ s.t. $\gamma . z \in \mathcal{F}$.

$$\mathrm{Im}(\gamma . z) = \frac{\mathrm{Im}(z)}{|cz+d|^2}$$

$$|cz+d|^2 = c^2|z|^2 + 2\,\mathrm{Re}(z)\,cd + d^2 > c^2 + d^2 - |cd|$$
$$= (|c|-|d|)^2 + |cd|.$$

$$(|c|-|d|)^2 + |cd| = 0 \Rightarrow c = d = 0 \quad ※ \quad \mapsto |cz+d|^2 > 1.$$

$\Rightarrow \operatorname{Im}(\gamma.z) < \operatorname{Im}(z)$

Similarly we have $\operatorname{Im}(z) = \operatorname{Im}(\gamma^{-1}.\gamma.z) \leq \operatorname{Im}(\gamma.z)$

which is a contradiction. ■.

- $\operatorname{vol}\left({}_{SL_2(\mathbb{Z})}\backslash SL_2(\mathbb{R})\right) = ?$

- What volume form are we using?

- $f: SL_2(\mathbb{R}) \to \mathbb{C} \rightsquigarrow \bar{f}: \mathcal{H} \to \mathbb{C}$,

$$\bar{f}(x+iy) = \int_K f(n(x)\, a(\sqrt{y})\, k)\, dk$$

$x+iy = \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix} . i = \begin{bmatrix} \sqrt{y} & x/\sqrt{y} \\ & \sqrt{y}^{-1} \end{bmatrix} . i = n(x)\, a(\sqrt{y}) . i$ where $n(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$

and $a(y) = \begin{bmatrix} y & \\ & y^{-1} \end{bmatrix}$.

Then $f \mapsto \int_{\mathcal{H}} \bar{f}(z)\, d_{\mathcal{H}} z$ defines

a left G-invariant measure (why?) (look at my note for the next lecture.)

Let $\tilde{\mathcal{F}} := \{ g \in SL_2(\mathbb{R}) \mid g_{22} \geq 0,\ g.i \in \mathcal{F} \}$, where

$\mathcal{F} := \{ z \in \mathcal{H} \mid |\operatorname{Re}(z)| \leq 1/2,\ |z| \geq 1 \}$.

<u>Lemma</u>. $\tilde{\mathcal{F}}$ is a fundamental domain of $SL_2(\mathbb{Z})$ in $SL_2(\mathbb{R})$, i.e.

① $SL_2(\mathbb{R}) = \bigcup_{\gamma \in SL_2(\mathbb{Z})} \gamma \tilde{\mathcal{F}}$.

② $\forall \gamma \in SL_2(\mathbb{Z}) \setminus I$, $\gamma \tilde{\mathcal{F}}^\circ \cap \tilde{\mathcal{F}} = \emptyset$

③ $\mu_1(\partial\widetilde{\mathcal{F}}) = 0$.

Pf. ① $\forall g \in SL_2(\mathbb{R})$, $\exists \gamma \in SL_2(\mathbb{Z})$ s.t. $\gamma . g . i \in \mathcal{F}$

$\Rightarrow$ either $\gamma g \in \widetilde{\mathcal{F}}$ or $-\gamma g \in \widetilde{\mathcal{F}}$.

$\Rightarrow$ $g \in \gamma^{-1}\widetilde{\mathcal{F}}$ or $(-\gamma)^{-1}\widetilde{\mathcal{F}}$.

② Suppose $g \in \widetilde{\mathcal{F}}^\circ$ and $\gamma g \in \widetilde{\mathcal{F}}$. So $\gamma g . i \in \mathcal{F}^\circ$ and $g . i \in \mathcal{F}$

$$\left.\begin{array}{l} \Rightarrow \gamma = \pm I \\ g_{22} > 0 \text{ and } (\gamma . g)_{22} \geq 0 \end{array}\right\} \Rightarrow \gamma = I.$$

③ Clear. ■

Corollary $\mu_1\left({}_{SL_2(\mathbb{Z})}\backslash^{SL_2(\mathbb{R})}\right) = \mu_1(\widetilde{\mathcal{F}})$.

Prop. $\mu_1\left({}_{SL_2(\mathbb{Z})}\backslash^{SL_2(\mathbb{R})}\right) = \frac{\pi^2}{3} = 2\,\zeta(2)$.

Pf. $\mu_1(\widetilde{\mathcal{F}}) = \int_{\mathcal{H}} \overline{\mathbb{1}}_{\widetilde{\mathcal{F}}}(z)\, d_{\mathcal{H}} z$

$$\overline{\mathbb{1}}_{\widetilde{\mathcal{F}}}(x+iy) = \int_0^{2\pi} \mathbb{1}_{\widetilde{\mathcal{F}}}(n(x)\, a(\sqrt{y})\, k(\theta))\, d\theta$$

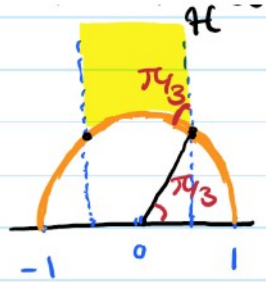
$$= \mathbb{1}_{\mathcal{F}}(x+iy) \int_0^{2\pi} \left[(n(x)\, a(\sqrt{y})\, k(\theta))_{22} \geq 0\right] d\theta$$

$$\boxed{n(x)\, a(y)\, k(\theta) = \begin{bmatrix} y & xy^{-1} \\ & y^{-1} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} * & * \\ * & y^{-1}\cos\theta \end{bmatrix}}$$

$$= \mathbb{1}_{\mathcal{F}}(x+iy) \cdot \pi$$

$$\Rightarrow \mu_1(\widetilde{\mathcal{F}}) = \pi \cdot \mathrm{vol}_{\mathcal{H}}(\mathcal{F}) = \frac{\pi^2}{3}$$

$\pi/3$



$$\mathrm{vol}_{\mathcal{H}}(\mathcal{F}) = \pi - \frac{\pi}{3} - \frac{\pi}{3}$$

$$= \frac{\pi}{3}. \qquad \blacksquare$$