Lecture 4: volume

Wednesday, September 30, 2015 9:05 AM

<u>Remark</u>. Hyperbolic geometry is a Riemannian geometry: we identify $T_z\mathcal{H}$ with $\mathbb{C}$ and $\langle v, w\rangle_z := \frac{1}{(\text{Im } z)^2}(v_1 w_1 + v_2 w_2)$. In particular, the angle between curves in Euclidean and hyperbolic geometries are the same.

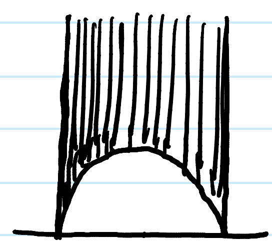
. Using the Riemannian structure, we get a volume form which is invariant under hyperbolic isometries:

$$\text{vol}_{\mathcal{H}}(A) := \int_A \frac{dx\,dy}{y^2}.$$

[volume form is $\sqrt{\det\left(\frac{1}{(\text{Im}(z))^2} I\right)}\, dx\,dy$]
$= \frac{dx\,dy}{y^2}$.

<u>Ex</u>. Area of <u>ideal triangles</u> is π.

<u>Solution</u> For any three points $\xi_1 < \xi_2 < \xi_3$ there is a Möbius transformation which sends it to $-1, 1, \infty$. So it is enough to consider this triangle:



$$\int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\infty} \frac{dy}{y^2}\, dx$$

$$= \int_{-1}^{1} \frac{-1}{y}\Big|_{\sqrt{1-x^2}}^{\infty} dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx \underset{x=\cos\theta}{=} \int_{\pi}^{0} \frac{1}{\sin\theta}(-\sin\theta)\, d\theta$$

$= \pi$.

<u>Ex</u>. Suppose T is a triangle with two vertices at infinity, and the other angle is α. Then $\text{area}_{\mathcal{H}}(T) = \pi - \alpha$.

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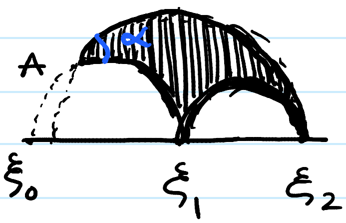
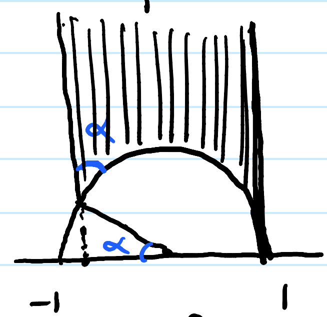
Solution



There is an isometry which sends ξ_0, ξ_1, ξ_2 to $-1, 1, \infty$. So we can assume T is of the following form



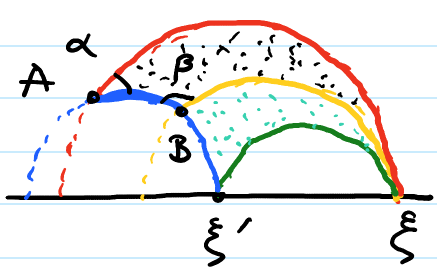
$$\int_{-\cos\alpha}^{1} \int_{\sqrt{1-x^2}}^{\infty} \frac{dy}{y^2}\, dx$$

$$= \int_{-\cos\alpha}^{1} \frac{-1}{y} \Big|_{\sqrt{1-x^2}}^{\infty} dx = \int_{-\cos\alpha}^{1} \frac{1}{\sqrt{1-x^2}}\, dx \underset{x = \cos\theta}{=} \int_{\pi-\alpha}^{0} -1\, d\theta$$

$$= \pi - \alpha. \quad \blacksquare$$

Ex. T: hyperbolic triangle, one ideal vertex and two angles are α and β. Then $\text{area}_{\mathcal{H}}(T) = \pi - \alpha - \beta$.

Solution.



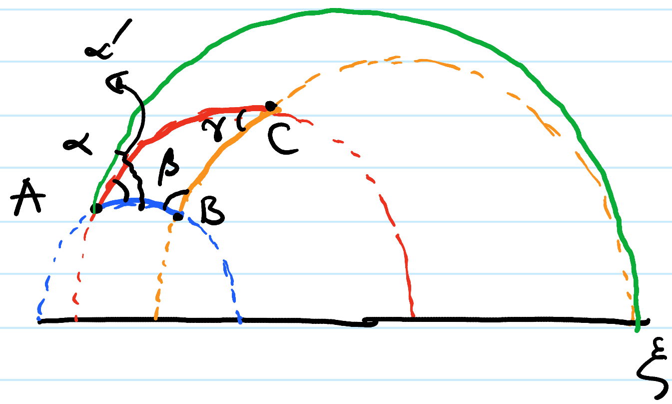
$$\text{area}_{\mathcal{H}}(T) = \text{area}_{\mathcal{H}}(\overset{\triangle}{A\xi\xi'}) - \text{area}_{\mathcal{H}}(\overset{\triangle}{B\xi\xi'})$$

$$= (\pi - \alpha) - (\pi - (\pi - \beta))$$

$$= \pi - \alpha - \beta. \quad \blacksquare$$

Thm (Gauss-Bonnet). Area of a hyperbolic triangle with angles

α, β, γ is $\pi-\alpha-\beta-\gamma$.



Pf.

$$\mathrm{area}_{\mathcal{H}} T = \mathrm{area}_{\mathcal{H}}(\overset{\triangle}{AB\xi}) - \mathrm{area}_{\mathcal{H}}(\overset{\triangle}{AC\xi})$$

$$= \left(\pi - \alpha' - \beta\right) - \left(\pi - (\alpha'' + \pi - \gamma)\right)$$

$$= \pi - (\alpha' - \alpha'') - \beta - \gamma = \pi - \alpha - \beta - \gamma. \quad \blacksquare$$

Let me quickly recall what a covering space of a topological space X is: $\tilde{X} \xrightarrow{p} X$ is called a <u>covering map</u> if

for any $x_0 \in X$, $\exists\, U_{x_0}$ a nbhd of x_0 s.t. $p^{-1}(U_{x_0})$ is a disjoint union $\bigsqcup O_i$ of open set s.t. for any i, $p|_{O_i} : O_i \to U_{x_0}$ is a homeomorphism.

For a topological space that is connected, path connected, and locally simply connected there is a universal covering space and $\mathrm{Aut}_X(\tilde{X}) := \{ \gamma : \tilde{X} \xrightarrow{\sim} \tilde{X} \mid p \circ \gamma = \gamma \}$ is called the group of <u>deck transformations</u>.

Clearly $\mathrm{Aut}_X(\tilde{X}) \curvearrowright \tilde{X}$. This action is <u>properly discontinuous</u>, i.e. and <u>Γ- orbit is locally finite</u>.

orientable

and Γ- orbit is locally finite.

Q. Let S be a (topological) orientable surface. Can we put a hyperbolic structure on S ?

There is the following classification result:

Thm. Any complete hyperbolic surface X is isometric to $\mathcal{H}/\Gamma$ where Γ is a torsion free discrete subgp of $\mathrm{Isom}(\mathcal{H})$.

Moreover $\mathcal{H}/\Gamma$ and $\mathcal{H}/\Delta$ are isometric if and only if Γ is a conjugate of Δ in $\mathrm{Isom}(\mathcal{H})$.

For a given orientable surface Σ_g of genus $g \geq 2$, let

$\mathcal{T}(\Sigma_g) :=$ possible hyperbolic structures on Σ_g

$$= \left\{ (X, f) \;\middle|\; \begin{array}{l} X : \text{hyperbolic surface} \\ f : X \xrightarrow{\sim} \Sigma_g \ \text{(marking)} \end{array} \right\} / \sim$$

where
$$\begin{tikzcd} X \arrow[rr, "\theta"] \arrow[dr, "f"'] & \arrow[d, phantom, "\circlearrowright"] & X' \arrow[dl, "f'"] \\ & \Sigma_g & \end{tikzcd}$$
(an isometry) gives us $(X, f) \sim (X', f')$.

One can see that this equivalent to

$$\left\{ \rho : \pi_1(\Sigma_g) \longrightarrow PSL_2(\mathbb{R}) \;\middle|\; \begin{array}{l} \cdot \ \mathrm{Im}\,\rho \text{ is discrete} \\ \cdot \ \rho \text{ is faithful} \end{array} \right\} /\!\!/ PSL_2(\mathbb{R})$$

(Character variety)

(up to conjugation.)

Notice that $\mathrm{Out}(\pi_1(\Sigma_g)) \curvearrowright \mathcal{T}(\Sigma_g)$.

Thm (Dehn–Nielsen–Baer) $\mathrm{Mod}^{\pm}(\Sigma_g) \simeq \mathrm{Out}(\underbrace{\pi_1(\Sigma_g)}_{\text{called the surface gp.}})$

where $\mathrm{Mod}^{\pm}(\Sigma_g) := \mathrm{Homeo}(\Sigma_g) / \mathrm{Homeo}(\Sigma_g)^{0}$

the homotopy classes of homeomorphisms of Σ_g.

The second point of view is the starting point of higher dimensional Teichmuller Theory which is an extremely active area of research:

$\{\rho : \Gamma \to G \mid \text{discrete, faithful}\} /\!/ G\text{-conj}.$