Lecture 3: Isometries, type of elements, and ping-pong players.

Wednesday, September 30, 2015 2:56 PM

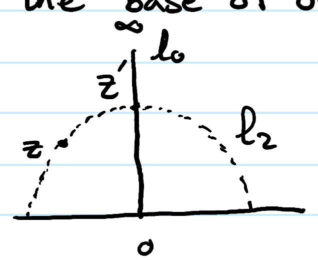
<u>Thm</u>. $\mathrm{Isom}(\mathcal{H}) = PSL_2(\mathbb{R}) \sqcup PSL_2(\mathbb{R})\tau$ where $\tau(z) = -\bar{z}$.

<u>Pf</u>. Let $\phi \in \mathrm{Isom}(\mathcal{H})$, and ℓ_0 be the geodesic connecting 0 to ∞. Then $\phi(\ell_0)$ is a hyperbolic geodesic. So there is $[g]$ in $PSL_2(\mathbb{R})$ s.t. $[g]\cdot\phi(\ell_0) = \ell_0$ and $[g]\cdot\phi(i) = i$. So $\Psi := [g]\circ\phi$ is an isometry which sends 0 to 0, ∞ to ∞, and i to i. Hence Ψ fixes ℓ_0 pointwise.

For any $z \in \mathcal{H}$, let ℓ_z be the geodesic orthogonal to ℓ_0 that passes through z, and let z' be the base of ortho. proj.

So $\Psi(z') = z'$, $\Psi(\ell_z) = \ell_z$

and $d_{\mathcal{H}}(z, z') = d_{\mathcal{H}}(\Psi(z), z')$.



So either $\Psi(z) = -\bar{z}$ or z. It is easy to show that this implies either $\Psi = \tau$ or id., and we are done. ∎

<u>Cor</u>. $\mathrm{Isom}(\mathcal{H})^{\circ} \simeq PSL_2(\mathbb{R})$.

$\overline{\mathcal{H}} = \mathcal{H} \cup \partial\mathcal{H}$ is a compact space, and any Möbius transformation has a fixed point in $\overline{\mathcal{H}}$.

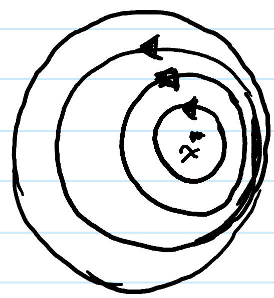
. If $\gamma\cdot p_1 = p_1$, $\gamma\cdot p_2 = p_2$, $p_1, p_2 \in \mathcal{H}$, then $\gamma\cdot p = p$ for any p in the geodesic which passes through p_1 and p_2. Since γ is orientation preserving, we have that $\gamma = id$.

- If γ fixes three points in $\partial\mathcal{H}$, then $\gamma = id$. (why?)
- If γ fixes p in $\mathcal{H}$ and ξ in $\partial\mathcal{H}$, then it fixes any point in the geodesic connecting p to ξ. So it is identity again.

So there are three possibilities for a non-trivial orientation preserving isometry (Möbius transformation).

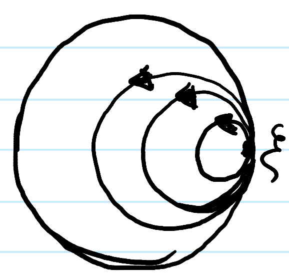
A single fixed point in $\mathcal{H}$: elleptic.

Conjugates of $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$



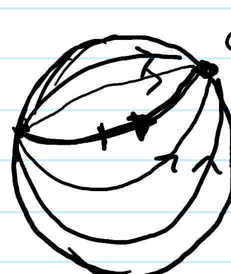
A single point in $\partial\mathcal{H}$: parabolic

Conjugates of $\begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}$



Two points in $\partial\mathcal{H}$: hyperbolic

Conjugates of $\begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix}$



(translation along a hyperbolic geodesic)

- If a is a hyperbolic element, then for any small nbhds U_a^+ and U_a^- of its contracting and repelling points we have

$a^n(\partial\mathcal{H} - U_a^-) \subset U_a^+$ and $a^{-n}(\partial\mathcal{H} - U_a^+) \subset U_a^-$

and U_a of its contracting and repelling points we have

$$a^n(\mathcal{H}\setminus U_a^-) \subseteq U_a^+ \quad \text{and} \quad a^{-n}(\mathcal{H}\setminus U_a^+) \subseteq U_a^-$$

for large enough $\underline{n}$.

<u>Thm</u> (Schottky) Suppose a and b are hyperbolic elements and $\mathrm{Fix}(a) \cap \mathrm{Fix}(b) = \emptyset$ (e.g. $a = \begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix}$ for some $\lambda > 1$ and $b = gag^{-1}$ for some $g \in SL_2(\mathbb{R}) \setminus \left(\left\{ \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \right\} \cup \left\{ \begin{bmatrix} * & 0 \\ * & * \end{bmatrix} \right\} \cup \left\{ \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \right\} \right)$),

Then, for large enough n, a^n and b^n generate a free group.

<u>Pf</u>. Let $\omega = (a^n)^{i_1} \omega'$ $(i_1 > 0)$ be a reduced word. In particular, ω' does NOT end with $\underline{a^{-n}}$.

For $x_0 \in \mathcal{H} \setminus (U_a^+ \cup U_a^- \cup U_b^+ \cup U_b^-)$ we have $\omega' \cdot x_0$ is not in U_a^-. So $(a^n)^{i_1} \cdot (\omega' \cdot x_0) \in U_a^+$.

In particular, $\omega \cdot x_0 \neq x_0 \Rightarrow \omega \neq \mathrm{id}$. ∎

We say a^n, a^{-n}, b^n, and b^{-n} are playing ping-pong.