Lecture 20: Mapping Γ-compact flats

Friday, March 10, 2017 9:07 AM

$\boxed{\text{Step 3}}$ $$hd(\phi(F_1),F_2) = \max\{\overbrace{\sup_{x\in F_1} d(\phi(x),F_2)}^{d_1}, \overbrace{\sup_{x\in F_2} d(\phi(F_1),x)}^{d_2}\}$$

$$d_1 = \sup_{x\in F_1} d(\phi(x),F_2) \quad \text{and} \quad d_2 = \sup_{x\in F_2} d(\phi(F_1),x).$$

- $$d_1 = \sup_{x\in F_1} d(\phi(x),F_2) = \sup_{x\in F_1} d(pr_{F_2}(\phi(x)),\phi(x))$$

$$\geq \sup_{x\in F_1} d(pr_{F_2}(\phi(x)),\phi(F_1)) = \sup_{x'\in F_2} d(x',\phi(F_1)) = d_2.$$

- $$N_{d_2}(\phi(F_1)) \supseteq F_2 \implies \sup_{x\in\phi^{-1}(F_2)} d(Pr_{F_1}(x),x) \leq \max\{b,kd_2\}$$

(again we are using surjectivity of ϕ.)

Since $pr_{F_1}(N_b(\phi^{-1}(F_2))) = F_1$, $\forall x_1 \in F_1$, $\exists x\in\phi^{-1}(F_2)$

and $x'\in X_1$ s.t. $d(x,x')\leq b$ and $pr_{F_1}(x') = x_1$.

$$\left.\begin{array}{l} \Rightarrow d(x_1, pr_{F_1}(x)) \leq b \\ d(pr_{F_1}(x),x) \leq \max\{b,kd_2\} \end{array}\right\} \Rightarrow d(x_1,x) \leq \max\{2b, b+kd_2\}.$$

$$\Rightarrow F_1 \subseteq N_{2\max\{b,kd_2\}}(\phi^{-1}(F_2))$$

$$\Rightarrow \phi(F_1) \subseteq N_{2k\max\{b,kd_2\}}(F_2)$$

$$\Rightarrow d_1 \leq 2k\max\{b,kd_2\}.$$

So we can assume $b < kd_2$, and we get $\boxed{d_2 \leq d_1 \leq 2k^2 d_2.}$

Lecture 20: Mapping Γ-compact flats

Tuesday, March 14, 2017 1:06 AM

Setting: $d_1 = \sup_{x_1 \in F_1} d(\Phi(x_1), F_2)$ and $d_2 = \sup_{x_2 \in F_2} d(\Phi(F_1), x_2)$.

We have $b < d_2 k$; $d_2 \le d_1 \ll d_2$.

The following is the key geometric information:

when we are away from a maximal flat F, the orthogonal projection pr_F causes more contraction.

<u>Key Geometric Fact</u> $F \subseteq X$ a maximal flat; $p \notin F$;

$\underset{\text{subspace}}{V \subseteq} T_p X$ and $\dim V = \dim F$.

Let $\tau : V \to T_{pr(p)} F$,

$\tau(v) := d\, pr|_p (v)$.

Then $|\det \tau| \ll_X d(p, F)^{-1/2}$.

<u>How are we going to use this?</u> We are going to cover a ball of radius $\Theta(d_2)$ in F_2 by orthogonal projection of a set that away from F_2 by distance at least $\Theta(d_2)$;

OK let's see the details:

Let $h(x_2) := d(\Phi(F_1), x_2)$. Then, $\forall \gamma \in \Delta$, $h(\gamma x_2) = h(x_2)$

as Φ is Γ-equiv. and F_1 is Δ-invariant. So h is a continu. function on the torus $\Delta \backslash F_2$. So $\exists\, x_2 \in F_2$ s.t. $d_2 = d(\Phi(F_1), x_2)$.

Since $pr_{F_2}(\Phi(F_1)) = F_2$, we can cover $B(x_2, d_2/2) \cap F_2$.

Suppose $y_2 \in \Phi(F_1)$ and $pr_{F_2}(y) \in B(x_2, d_2/2)$. Then

<u>Claim</u>. $y_2 \in B(x_2, \Theta(d_2)) \setminus N_{d_2/2}(F_2)$.

<u>Pf of claim</u> ① $d(x_2, y_2) \le d(x_2, pr_{F_2}(y_2)) + d(pr_{F_2}(y_2), y_2)$

$$\ll d_2 \quad \text{as} \quad \Phi(F_1) \subseteq N_{\Theta(d_2)}(F_2).$$

② $$d(y_2, F_2) = d(y_2, pr_{F_2}(y_2)) \ge d(x_2, y_2) - d(x_2, pr_{F_2}(y_2))$$
$$\ge d_2 - d_2/2 = d_2/2. \quad \blacksquare$$

Hence $pr_{F_2}(\Phi(F_1) \cap B(x_2, \Theta(d_2)) \setminus N_{d_2/2}(F_2)) \supseteq B(x_2, d_2/2) \cap F_2$.

$\Rightarrow$ By the mentioned geometric fact, we have

$$\mathrm{vol}(\Phi(F_1) \cap B(x_2, \Theta(d_2)) \setminus N_{d_2/2}(F_2)) \gg d_2^{1/2}\, \mathrm{vol}(B(x_2, d_2/2) \cap F_2)$$
$$\gg d_2^{1/2 + r_0}.$$

As Φ is k-Lipschitz and QI, $\mathrm{vol}(\Phi(F_1) \cap B(x_2, \Theta(d_2))) \ll \mathrm{vol}(F_1 \cap B(x', \Theta(d_2)))$

$$\ll d_2^{r_0}.$$

$$\Rightarrow d_2^{1/2} \ll 1 \Rightarrow d_2 \ll 1. \quad \blacksquare$$

Lecture 20: Γ-equivariant map between the space of flats

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Now we want to continuously extend $\overline{\Phi}: \mathcal{F}_{1,\Gamma} \to \mathcal{F}_{2,\Gamma}$ to

$\overline{\Phi}: \mathcal{F}_1 \to \mathcal{F}_2$ s.t. $hd(\overline{\Phi}(F_1), \phi(F_1)) \underset{(\lambda,C)}{\ll} 1$.

To do so it is enough to show

(*) $\boxed{\text{If } F_i^{(1)} \in \mathcal{F}_{1,\Gamma} \text{ and } F_i^{(1)} \longrightarrow F^{(1)}, \text{ then } \overline{\Phi}(F_i^{(1)}) \text{ converges to a flat } F^{(2)}.}$

(As $\mathcal{F}_{1,\Gamma}$ is dense in $\mathcal{F}_1$, for any $F^{(1)} \in \mathcal{F}_1$, $\exists F_i \to F^{(1)}$ and $F_i^{(1)} \in \mathcal{F}_{1,\Gamma}$. Then let $\overline{\Phi}(F^{(1)}) := \lim_{i\to\infty} \overline{\Phi}(F_i^{(1)})$

[(*) implies the existence of this limit and its independence on the choice of $\{F_i^{(1)}\}$.]

• $\forall x \in F^{(1)}$, $\exists x_i \in F_i^{(1)}$ s.t. $d(x, x_i) \ll 1$

$\left.\begin{array}{l} \Rightarrow d(\phi(x), \phi(x_i)) \ll 1. \\ \exists\, y_i \in \overline{\Phi}(F_i^{(1)}) \text{ s.t. } d(\phi(x_i), y_i) \ll 1 \end{array}\right\} \Rightarrow$

passing to a subseq. $y_i \longrightarrow y \in \overline{\Phi}(F^{(1)})$

$\Rightarrow d(\phi(x), y) \ll 1. \Rightarrow \phi(x) \in N_{\Theta(1)}(\overline{\Phi}(F^{(1)}))$.

• $\left.\begin{array}{l} y \in \overline{\Phi}(F^{(1)}) \Rightarrow \exists\, y_i \in \overline{\Phi}(F_i^{(1)}) \text{ s.t. } y_i \to y \\ \Rightarrow \exists\, x_i \in F_i^{(1)} \text{ s.t. } d(\phi(x_i), y_i) \ll 1. \end{array}\right\} \Rightarrow \{x_i\}$ bounded

Lecture 20: Γ-equivariant map between the space of flats

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Passing to a subseq. $x_i \to x$. $\Rightarrow x \in F^{(1)}$ and $\phi(x_i) \to \phi(x)$.

$\Rightarrow d(y, \phi(x)) \ll 1 \rightsquigarrow hd(\Phi(F^{(1)}), \phi(F^{(1)})) \ll 1$.)

- For $s > b$, let $\mathcal{F}_s^{(2)} := \{ F \in \mathcal{F}_2 \mid hd(\phi(F^{(1)}) \cap B_s^{(2)}, F \cap B_s^{(2)}) \leq 2 \underbrace{c}_{\text{the above implied constant}} \}$

- $\mathcal{F}_s^{(2)}$ is a compact subset of $\mathcal{F}^{(2)}$.

- Since $F_i^{(1)} \to F^{(1)}$, we have $hd(F_i^{(1)} \cap B_{ks}^{(1)}, F^{(1)} \cap B_{ks}^{(1)}) \to 0$

$\Rightarrow hd(\phi(F_i^{(1)}) \cap B_s^{(2)}, \phi(F^{(1)}) \cap B_s^{(2)}) \to 0$

$\Rightarrow hd(\Phi(F_i^{(1)}) \cap B_s^{(2)}, \phi(F^{(1)}) \cap B_s^{(2)}) \leq 2c$ if $i \gg 1$.

$\Rightarrow \mathcal{F}_s^{(2)} \neq \emptyset$. $\Rightarrow \bigcap_{s>b} \mathcal{F}_s^{(2)} \neq \emptyset$.

$\forall F^{(2)} \in \bigcap_{s>b} \mathcal{F}_s^{(2)}$ we have $hd(F^{(2)}, \phi(F^{(1)})) \ll 1$.

$\Rightarrow$ there is only one element in this intersection

as $hd(F^{(2)}, F^{(2)\prime}) \ll 1 \Rightarrow F^{(2)} = F^{(2)\prime}$.

Lecture 20: The space of flats are homeomorphic

Tuesday, March 14, 2017 10:50 AM

- $\underline{\overline{\Phi} \text{ is injective}}$ If $\overline{\Phi}(F_1) = \overline{\Phi}(F_1')$, then

$$hd(\Phi(F_1), \Phi(F_1')) \ll 1 \Rightarrow hd(F_1, F_1') \ll 1$$

$$\Rightarrow F_1 = F_1'.$$

- $\underline{\overline{\Phi} \text{ is a homeomorphism.}}$

Let $\Phi' : X_2 \to X_1$ be a Γ-equivariant with similar properties as Φ. And define $\overline{\Phi}'$. Notice that $\forall\, F_1 \in \mathcal{F}_{1,\Gamma}$, $\overline{\Phi}(F_1)$ was the unique flat which was stabilized by $\Delta \subseteq \Gamma \cap G_{F_1}$.

$$\Rightarrow \overline{\Phi}'(\overline{\Phi}(F_1)) = F_1.$$

$$\Rightarrow \overline{\Phi}' \circ \overline{\Phi}\big|_{\mathcal{F}_{1,\Gamma}} = \text{identity} \Rightarrow \overline{\Phi}' \circ \overline{\Phi} \text{ is identity}$$

So $\overline{\Phi}$ is a homeomorphism. ∎

Lecture 20: Why are flats important: Tits spherical building

Thursday, March 16, 2017 11:56 AM

To any semisimple algebraic group G defined over a field k, Tits attached a spherical building:

Let $T(G,k)$ be the set of all the parabolic k-subgps.

for any maximal k-split k-torus $S \subseteq G$, let Σ_S be the (finite) subset of $T(G,k)$ which consists of those parabolics that contain S. Such Σ_S is called an apartment. Let $\mathcal{A} := \{\Sigma_S \mid S: \text{max. } k\text{-split } k\text{-torus}\}$.

- <u>Theorem (Tits)</u> Suppose G has no simple factor of k-rank ≤ 1, and $Z(G) = 1$. Then $(T(G,k), \mathcal{A})$ uniquely determines $G(k)^+ := \langle R_u(P)(k) \mid P \in T(G,k) \rangle$.
- Furthermore Tits proved that $(T(G,k), \subseteq)$ determines $\mathcal{A}$ if G has no simple factor of k-rank ≤ 1.
- With our control of on flats, we will control the structure of the spherical building which enables us to prove strong rigidity.