# Lecture 17: Quasi-isometric embedding

Tuesday, March 7, 2017 8:36 AM

<u>Def.</u> Let $(X_1, d_1)$ and $(X_2, d_2)$ be two metric spaces. Then

① $\phi: X_1 \to X_2$ is called a <u>$(\lambda, C)$-quasi-isometric embedding</u> if

$$\forall x,y \in X_1, \ \frac{1}{\lambda} d(x,y) - C \leq d_2(\phi(x), \phi(y)) \leq \lambda\, d(x,y) + C$$

② A $(\lambda, C)$-quasi-isometric embedding $X_1 \xrightarrow{\phi} X_2$ is called <u>$(\lambda, C)$-quasi-isometry</u> if $N_C(\phi(X_1)) = X_2$.

<u>Ex.</u> Suppose $\Omega_1$ and $\Omega_2$ are two finite symmetric generating sets of $\Gamma$. Then $\phi: Cay(\Gamma, \Omega_1) \to Cay(\Gamma, \Omega_2)$,

$\phi(\gamma) = \gamma$ is a quasi-isometry.

<u>Pf.</u> $\Omega_1 \subseteq B_{\Omega_2}(r_0)$ and $\Omega_2 \subseteq B_{\Omega_1}(r_0)$ for some $r_0 \geq 1$. So

$$d_1(e, \gamma) \leq r_0\, d_2(e, \gamma) \quad \text{and} \quad d_2(e, \gamma) \leq r_0\, d_1(e, \gamma)$$

$$\Rightarrow \frac{1}{r_0} d_1(x,y) \leq d_2(\phi(x), \phi(y)) \leq r_0\, d_1(x,y). \quad \square$$

<u>Def.</u> We say $\phi \sim \psi$ for $X \xrightarrow{\phi} Y$ and $X \xrightarrow{\psi} Y$ if $\exists\, c \in \mathbb{R}^+$ s.t. $\forall x \in X$, $d(\phi(x), \psi(x)) \leq c$.

<u>Exercise</u> $\sim$ is an equivalence relation.

# Lecture 17: Quasi-isometries

Thursday, March 2, 2017 8:54 AM

A few basic properties:

- $\left.\begin{array}{l}\phi : X \to Y \text{ is QI} \\ \phi \sim \psi\end{array}\right\} \Rightarrow \psi$ is QI. ; so we can talk about QI class $[\phi]$ of functions.
- $X \xrightarrow{\phi} Y$ and $Y \xrightarrow{\psi} Z$ are QIs $\Rightarrow \psi \circ \phi$ is QI.
- $X \xrightarrow{\phi_i} Y$ and $Y \xrightarrow{\psi_i} Z$ are QIs, $\phi_1 \sim \phi_2$, $\psi_1 \sim \psi_2 \Rightarrow$ $\psi_1 \circ \phi_1 \sim \psi_2 \circ \phi_2$. So $[\psi] \cdot [\phi] := [\psi \circ \phi]$ is well-defined.
- $X \xrightarrow{\phi} Y$ is QI $\Rightarrow \exists\, \psi : Y \to X$ which is QI and
$$[\psi][\phi] = [id_X] \text{ and}$$
$$[\phi][\psi] = [id_X].$$
- $QI(X) := \{\phi : X \to X \mid \phi : \text{quasi-isom}\}/\sim$ is a group.

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<u>The Švarc-Milnor lemma</u>. $X$: geodesic space. $\Gamma \curvearrowright X$ properly and cocompactly by isometries. Then

① $\Gamma$ is finitely generated

② for any $x_0 \in X$, $\gamma \mapsto \gamma \cdot x_0$ is a QI from a (locally finite) Cayley graph of $\Gamma$ to $X$.

<u>Proof</u>. Let $B(x_0, r_0)$ be the closed ball in $X$ s.t. $\Gamma B(x_0, r_0) = X$.

# Lecture 17: Svarc-Milnor lemma

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Since $\Gamma \curvearrowright X$ is proper, $\Omega := \{\gamma \in \Gamma \mid \gamma B(x_0, 3r_0) \cap B(x_0, 3r_0) \neq \emptyset\}$

is a finite (symmetric) set.

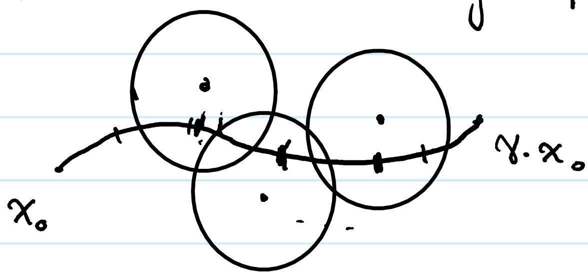
<u>Claim</u>. $\Gamma = \langle \Omega \rangle$.

<u>Pf of claim</u>. $\forall \gamma \in \Gamma$, let $L$ be a path connecting $x_0$ to

$\gamma . x_0$ with length $\leq d(x_0, \gamma . x_0) + 1$. So $L$ is the image of

$c : [0,1] \longrightarrow X$,

$c(0) = x_0$ and $c(1) = \gamma . x_0$




So $\exists$ $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = 1$ s.t. length of the piece

$\gamma_0 = id$, $\gamma_m = \gamma$

of curve connecting $c(t_i)$ to $c(t_{i+1})$ is $r_0$ for $i \leq m-2$, and

length of the piece of the curve connecting $c(t_{m-1})$ to $c(t_m)$ is $\leq r_0$.

In particular, $d(c(t_i), c(t_{i+1})) \leq r_0$, for any $i$.

$\Rightarrow \exists$ $\gamma_i \in \Gamma$ s.t. $c(t_i) \in \gamma_i B(x_0, r_0) = B(\gamma_i \cdot x_0, r_0)$

$$\Rightarrow d(\gamma_i \cdot x_0, \gamma_{i+1} \cdot x_0) \leq d(\gamma_i \cdot x_0, c(t_i)) + d(c(t_i), c(t_{i+1}))$$

$$+ d(\gamma_{i+1} \cdot x_0, c(t_{i+1})) \leq r_0 + r_0 + r_0 = 3r_0.$$

$$\Rightarrow d(x_0, \gamma_i^{-1} \gamma_{i+1} x_0) \leq 3r_0 \Rightarrow \emptyset \neq \gamma_i^{-1} \gamma_{i+1} B(x_0, 3r_0) \cap B(x_0, 3r_0)$$

$$\Rightarrow \gamma_i^{-1} \gamma_{i+1} \in \Omega \Rightarrow \gamma = \gamma_0 \cdot (\gamma_0^{-1} \gamma_1) \cdot (\gamma_1^{-1} \gamma_2) \cdot \cdots \cdot (\gamma_{m-1}^{-1} \gamma_m)$$

$$\in \underbrace{\Omega \cdot \Omega \cdot \cdots \cdot \Omega}_{m \text{ times}}.$$

# Lecture 17: Svarc-Milnor lemma

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The above argument also gives us that

$$d_\Omega(e,\gamma) \le m \le \text{length of } L/r_0 + 1$$
$$\le \frac{d(x_0,\gamma\cdot x_0)+1}{r_0} + 1 = \frac{1}{r_0} d(x_0,\gamma\cdot x_0) + \frac{1}{r_0} + 1 .$$

Now let $\lambda := \max\{d(x_0,\omega\cdot x_0) \mid \omega\in\Omega\} \le 6r_0$. Then

$$\begin{aligned} d(x_0,\gamma\cdot x_0) &= d(x_0, \omega_{i_1}\cdots\omega_{i_\ell}\cdot x_0) \\ &\le d(\omega_{i_1}\cdots\omega_{i_\ell}\cdot x_0, \omega_{i_1}\cdot\omega_{i_2}\cdots\omega_{i_{\ell-1}}\cdot x_0) \\ &\quad + d(\omega_{i_1}\cdots\omega_{i_{\ell-1}}\cdot x_0, \omega_{i_1}\cdots\omega_{i_{\ell-2}}\cdot x_0) \\ &\quad + \cdots + d(\omega_{i_1}\cdot x_0, x_0) \\ &= d(\omega_{i_\ell}\cdot x_0, x_0) + d(\omega_{i_{\ell-1}}\cdot x_0, x_0) + \cdots + d(\omega_{i_1}\cdot x_0, x_0) \\ &\le \lambda\, \ell = \lambda\, d_\Omega(e,\gamma). \end{aligned}$$

So $\gamma \mapsto \gamma\cdot x_0$ is a quasi-isometric embedding.

Since $N_{2r_0}(\Gamma\cdot x_0) = \Gamma\cdot \mathring{B}_{2r_0}(x_0) = X$, we get that

$\gamma \mapsto \gamma\cdot x_0$ is a QI. ■

<u>Mostow's strong rigidity</u>. Let $G_1$ and $G_2$ be two semisimple groups with trivial center and no compact simple factors. Suppose $G_i \not\simeq SL_2(\mathbb{R})$. Let $\Gamma_i \subseteq G_i$ be cocompact lattices.

# Lecture 17: Statement of Mostow's strong rigidity

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Suppose $\theta : \Gamma_1 \xrightarrow{\sim} \Gamma_2$. Then $\exists$ an analytic isomorphism

$\overline{\theta} : G_1 \xrightarrow{\sim} G_2$ s.t. $\overline{\theta}\big|_{\Gamma_1} = \theta$.

__A corollary of Švarc–Milnor lemma__

$\Gamma \subseteq G$ is a cocompact lattice in a semisimple group $G$

$$\Rightarrow \quad \begin{aligned} \Gamma &\xrightarrow{\Phi} X := G/K \\ \gamma &\longmapsto \gamma \cdot x_0 \end{aligned} \quad \text{is a QI.}$$

- So in the setting of Mostow's strong rigidity we get that

$$\begin{aligned} \Gamma &\xrightarrow{\Phi_1} X_1 \\ \gamma &\longmapsto \gamma \cdot x_1 \end{aligned} \quad \text{and} \quad \begin{aligned} \Gamma &\xrightarrow{\Phi_2} X_2 \\ \gamma &\longmapsto \theta(\gamma) \cdot x_2 \end{aligned} \quad \text{are QI s}$$

So $\exists\ \Psi_1 : X_1 \to \Gamma$ which is a quasi-inverse of $\Phi_1$

$\Rightarrow \quad \Phi := \Phi_2 \circ \Psi_1 : X_1 \longrightarrow X_2$ is a QI.

and $\Phi(\gamma \cdot x_0) = \theta(\gamma) \cdot x_0$.

In fact we can choose $\Phi$ carefully to get a $\Gamma$-equivariant QI.

__Proposition.__ In the setting of main theorem, if $\Gamma_i$'s are torsion-free, $\exists\ \Phi : X_1 \to X_2$ s.t. ① QI, ② $\Gamma$-equivariant, ③ continuous

④ $d(\Phi(x), \Phi(y)) \le \lambda\, d(x,y)$.