Lecture 16: Metric definition of maximal boundary

Tuesday, February 28, 2017 11:35 AM

So $\forall p \in P({}^{\triangleleft}A_0)$, $\{a^{-1}pa \mid a \in {}^{\triangleleft}A_0\}$ is bounded.

Let ${}^{\triangleleft}F_0 := {}^{\triangleleft}A_0 . x_K$. Then

$$hd(p\,{}^{\triangleleft}F_0, {}^{\triangleleft}F_0) \le \sup_{a \in {}^{\triangleleft}A_0} d(p a x_K, a x_K)$$
$$= \sup_{a \in {}^{\triangleleft}A_0} d(a^{-1}pa \cdot x_K, x_K)$$
$$< \infty .$$

- For any $k \in K$ s.t. $k\,{}^{\triangleleft}F_0 \neq {}^{\triangleleft}F_0$. So $\exists\, y \in {}^{\triangleleft}F_0$ s.t. $k.y \notin {}^{\triangleleft}F_0$. So the geodesic ray which connects x_K to y is sent to a geodesic ray which connects x_K to $k.y$. After taking log we get a half-line and a cone, with a common vertex (0). So using the Euclidean distance we get $hd(k\,{}^{\triangleleft}F_0, {}^{\triangleleft}F_0) = \infty$.
- If $k\,{}^{\triangleleft}F_0 = {}^{\triangleleft}F_0$, then $k\,{}^{\triangleleft}A_0 K = {}^{\triangleleft}A_0 K$
  $\Rightarrow k\,{}^{\triangleleft}A_0\, k^{-1} = {}^{\triangleleft}A_0 \Rightarrow k \in C_G({}^{\triangleleft}A_0) \Rightarrow k \in P({}^{\triangleleft}A_0)$.
- $hd(pk\,{}^{\triangleleft}F_0, {}^{\triangleleft}F_0) < \infty \Rightarrow hd(pk\,{}^{\triangleleft}F_0, p\,{}^{\triangleleft}F_0) - hd(p\,{}^{\triangleleft}F_0, {}^{\triangleleft}F_0) < \infty$
  $\Rightarrow hd(k\,{}^{\triangleleft}F_0, {}^{\triangleleft}F_0) < \infty \Rightarrow k \in P({}^{\triangleleft}A_0)$
- So $g\,{}^{\triangleleft}F_0 \sim {}^{\triangleleft}F_0 \Leftrightarrow g \in P({}^{\triangleleft}A_0)$. (why are we done?) $\square$

Lecture 16: Metric definition of maximal boundary

Tuesday, February 28, 2017 11:59 AM

<u>Corollary</u>. Suppose ${}^{\triangleleft}F_1$ and ${}^{\triangleleft}F_2$ are two chamber (wall)s in X of origins x_1, x_2 s.t. $hd({}^{\triangleleft}F_1, {}^{\triangleleft}F_2) < \infty$. Then

$$hd({}^{\triangleleft}F_1, {}^{\triangleleft}F_2) = d(x_1, x_2).$$

<u>Pf.</u> Let ℓ be a half-line in ${}^{\triangleleft}F_1$ which starts at x_1.

$\Rightarrow$ $x \longmapsto d(x, {}^{\triangleleft}F_2)$ is convex and bounded (on ℓ)

$\Rightarrow$ $d(\ell, {}^{\triangleleft}F_2) = d(x_1, {}^{\triangleleft}F_2)$ $\Rightarrow$ $\forall x \in {}^{\triangleleft}F_1$, $d(x_1, {}^{\triangleleft}F_2) \geq d(x, {}^{\triangleleft}F_2)$.

Similarly we get $\forall x \in {}^{\triangleleft}F_2$, $d(x_2, {}^{\triangleleft}F_1) \geq d(x, {}^{\triangleleft}F_1)$. So

$$hd({}^{\triangleleft}F_1, {}^{\triangleleft}F_2) = d(x_1, x_2). \qquad \blacksquare$$

Lecture 16: The displacement function

Thursday, March 2, 2017 8:21 AM

<u>Def.</u> The displacement of g is defined to be

$$d_g := \inf_{x\in X} d(x, gx).$$

① $d_g = d_{hgh^{-1}}$.

<u>Pf.</u> $d_{hgh^{-1}} = \inf_{x\in X} d(x, hgh^{-1}x) = \inf_{x\in X} d(h^{-1}x, g\, h^{-1}x)$

$$= \inf_{x'\in X} d(x', gx') = d_g. \qquad \blacksquare$$

② If $Y \subseteq X$ is a geodesic subspace and $gY = Y$, then

$$d_g = \inf_{y\in Y} d(y, gy)$$

<u>Pf.</u> $\forall x\in X$, $\left.\begin{array}{l}[x, \pi_Y(x)] \perp Y \\ gY = Y\end{array}\right\} \Rightarrow [gx, g(\pi_Y(x))] \perp Y$

$$\Rightarrow \pi_Y(gx) = g\pi_Y(x).$$

$$\Rightarrow d(x, gx) \geq d(\pi_Y(x), \pi_Y(gx)) = d(\pi_Y(x), g\pi_Y(x))$$

$$\Rightarrow d_g = \inf_{y\in Y} d(g, gy). \qquad \blacksquare$$

③ If $p \in P$, then $d_p = d(x_K, p x_K)$.

<u>Pf.</u> Let L be the geodesic passing through x_K and $p x_K$; then we know $t \longmapsto p^t x_K$ is a parametrization of this geodesic. So $pL = L$. Hence $d_p = \inf_{y\in L} d(y, py)$. On L, p is just a translation. So $d_p = d(x_K, p x_K)$.

Lecture 16: The displacement function

Thursday, March 2, 2017 8:43 AM

④ Notice that in our setting $X \subseteq P(n)$ (The symm. space of $SL_n(\mathbb{R})$) as a geodesic subspace. So, by ②, it is enough to understand the displacement of g as an element of $SL_n(\mathbb{R})$.

⑤ Let s be a semisimple element, and let $s=pk$ be its polar decomposition. Then $d_s = d_p$. And, if $p \in P$, $k \in K$, then $d_s = d(x_K, p x_K)$.

Pf. After conjugation in $SL_n(\mathbb{R})$, we can assume p is diagonal and $k \in SO(n)$. Let L be the geodesic $t \mapsto p^t \cdot x_K$. Then $s \cdot (p^t \cdot x_K) = (p^t s) \cdot x_K = p^{t+1} \cdot x_K \in L \Rightarrow sL = L$. So $d_s = \inf_{t \in \mathbb{R}} d(p^t \cdot x_K, p^{t+1} \cdot x_K)$
$= d(x_K, p \cdot x_K) = d_p$. ∎

⑥ For any $g \in SL_n(\mathbb{R})$, let $g = su$ be its Jordan decomposition. Then $d_{su} \leq d_s$.

Pf. As we have discussed it earlier, we have that $\exists\, h_i \in C_{SL_n(\mathbb{R})}(s)$ s.t. $h_i u h_i^{-1} \longrightarrow I$.

$$
\begin{aligned}
\inf_{x \in X} d(x, su\,x) &\leq \inf_i d(h_i^{-1} x_0, su\, h_i^{-1} x_0) \\
&= \inf_i d(x_0, s\, h_i u h_i^{-1} x_0) \\
&\leq \lim_{i \to \infty} d(x_0, s\, h_i u h_i^{-1} x_0) \\
&= d(x_0, s x_0) \quad \Rightarrow d_{su} \leq d_s. \quad ∎
\end{aligned}
$$

Lecture 16: The displacement function

Thursday, March 2, 2017 10:36 AM

⑦ Suppose $p \in \mathcal{P}(n)$, u is unipotent, and $pu = up$. Then

$$d(x_K, pux_K)^2 = d(x_K, px_K)^2 + d(x_K, ux_K)^2.$$

Pf. $d(x_K, g x_K)^2 = d(x_K, \sqrt{gg^t}\, x_K)^2$

$$= \mathrm{Tr}\left(\left(\log \sqrt{gg^t}\right)^2 \right) \quad \text{for any } g \in SL_n(\mathbb{R}).$$

We have to show

$$\mathrm{Tr}\,(\log(puu^t p))^2 = \mathrm{Tr}\,(\log p^2)^2 + \mathrm{Tr}\,(\log uu^t)^2.$$

After conjugation we can assume

$$p = \mathrm{diag}(\lambda_1 I_{n_1}, \dots, \lambda_m I_{n_m}) \text{ and } u = \mathrm{diag}(u_1, \dots, u_m)$$

where u_i is $n_i \times n_i$ – unipotent matrix.

So $\mathrm{Tr}\left(\log(puu^t p)^2\right) = \sum_{i=1}^m \mathrm{Tr}\left(\log(\lambda_i^2\, u_i u_i^t)^2\right)$

and $\mathrm{Tr}\left((\log p^2)^2\right) = \sum_{i=1}^m \mathrm{Tr}\left((\log \lambda_i^2 I_{n_i})^2\right)$

and $\mathrm{Tr}\left((\log uu^t)^2\right) = \sum_{i=1}^m \mathrm{Tr}\,(\log u_i u_i^t)^2.$

Hence, it is enough to observe $\det u_i u_i^t = 1$ and

$$\mathrm{Tr}(\log(\lambda^2 X)^2) = \mathrm{Tr}\left((\log \lambda^2 I + \log X)^2\right)$$

$$= \mathrm{Tr}\left((\log \lambda^2 I)^2\right) + \mathrm{Tr}(\log X)^2 + 2 \log \lambda^2\, \mathrm{Tr}(\log X)$$

$$= \mathrm{Tr}\left((\log \lambda^2 I)^2\right) + \mathrm{Tr}(\log X)^2 + 2 \log \lambda^2 \log(\det X). \quad \blacksquare$$

Lecture 16: The displacement function

Thursday, March 2, 2017 10:53 AM

(8) Suppose $g = pku$, $p \in \mathcal{P}(n)$, $k \in SO(n)$, and u unipotent, $s = pk = kp$ and su is the Jordan decomposition of g. Then $d_g = d_p$.

<u>Pf</u>. Let $H := C_{SL_n(\mathbb{R})}(s)$. Since $p = p^t$, $k^t = k^{-1}$, and $C(s) = C(p) \cap C(k)$, we have H is self-adjoint. So $H \cdot x_K$ is a geodesic subspace of $\mathcal{P}(n)$, which is pu-invariant.

Hence
$$d_g^2 = \inf_{h \in H} d(h \cdot x_K, pkuh \cdot x_K)^2 = \inf_{h \in H} d(x_K, p h^{-1} uh \cdot x_K)^2$$
$$= \inf_{h \in H} d(x_K, p \cdot x_K)^2 + d(x_K, h^{-1}uh \cdot x_K)^2$$
$$\geq d(x_K, p\, x_K)^2.$$

$\Rightarrow d_g \geq d_p$. Hence, by (5) and (6), $d_g = d_p$. ■

(9) In the setting of <u>8</u>, if $d(x, g\,x) = d_g \Rightarrow g$ is semisimple.

<u>Pf</u>. $\exists\, h \in C_G(s)$ s.t. $x = h \cdot x_K$. So
$$d_g^2 = d(h \cdot x_K, gh\, x_K)^2 = d(x_K, p\, h^{-1}uh\, x_K)^2$$
$$= d(x_K, p x_K)^2 + d(x_K, h^{-1}uh\, x_K)^2 \Rightarrow h^{-1}uh\, x_K = x_K$$
$\Rightarrow u = 1$. ■

Lecture 16: The displacement function

Thursday, March 2, 2017 11:28 AM

<u>Theorem.</u> $d_g^2 = d_{\mathrm{pol}\, g}^2 = 4\,\mathrm{Tr}\left((\log \mathrm{pol}\, g)^2\right)$; and

$\exists x \in X,\ d(x, gx) = d_g \iff g$ is semisimple.