Lecture 15: A bit of structure theory of semisimple groups

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Def.. $A \subseteq GL_n(\mathbb{R})$ is called a polar subgroup if

① A is connected. ② A is diagonalizable over $\mathbb{R}$.

. In the language of algebraic groups:

$A = S(\mathbb{R})^\circ$ where S is an $\mathbb{R}$-split $\mathbb{R}$-torus.

Theorem. If A_1 and A_2 are two maximal polar subgp of G, then $\exists\, g \in G$ s.t. $g A_1 g^{-1} = A_2$.

. Let A be a maximal polar subgroup.;

. Let $\mathfrak{g} = \operatorname{Lie} G := \{Y \in \mathfrak{gl}_n(\mathbb{R}) \mid \exp(tY) \in G \ \ \forall t \in \mathbb{R}\}$.

$\Rightarrow \forall g \in G, x \in \mathfrak{g}, \ \operatorname{Ad}(g)(x) := g x g^{-1} \in \mathfrak{g}$.

. $\operatorname{Ad}(A) \subseteq \operatorname{End}(\mathfrak{g})$ can be diag. / $\mathbb{R}$. So

$\mathfrak{g} = \bigoplus \mathfrak{g}_\alpha$ for some $\alpha \in \operatorname{Hom}(A, \mathbb{R}^+)$ where

$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid \operatorname{Ad}(a)(x) = \alpha(a)\, x\}$.

. $x_\alpha \in \mathfrak{g}_\alpha \Rightarrow \operatorname{Ad}(a)[x_\alpha, x_\beta] = [\operatorname{Ad}(a)(x_\alpha), \operatorname{Ad}(a)(x_\beta)]$

$= \alpha(a)\beta(a)\, [x_\alpha, x_\beta]$.

We often denote $\operatorname{Hom}(A, \mathbb{R}^+)$ additively; so

$[x_\alpha, x_\beta] \in \mathfrak{g}_{\alpha+\beta} \Rightarrow [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.

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. Let A be a maximal abelian subgroup of $\mathbb{P} := \mathbb{P}(n) \cap G$.

. Then $\mathrm{Ad}(A) \subseteq \mathrm{Aut}(\mathfrak{g})$ is diagonalizable where

$$\mathfrak{g} = \mathrm{Lie}(G) = \{ x \in M_n(\mathbb{R}) \mid \exp(tx) \in G \text{ for any } t \in \mathbb{R} \}.$$

and $\mathrm{Ad}(g)(x) = g x g^{-1}$.

. So $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \mathrm{Hom}(A, \mathbb{R}^+)} \mathfrak{g}_\alpha$ where

$$\mathfrak{g}_\alpha := \{ x \in \mathfrak{g} \mid \forall a \in A,\ \mathrm{Ad}(a)(x) = \alpha(a)\, x \}.$$

. Let $\Phi := \{ \varphi \in \mathrm{Hom}(A, \mathbb{R}^+) \mid \mathfrak{g}_\varphi \neq 0 \ ;\ \varphi \neq 0 \}$.

- $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.
- If $\alpha, \beta, \alpha+\beta \in \Phi$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$.
- $\mathfrak{g}_\alpha$ is nilpotent.

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- $\mathfrak{g}_0 = \mathrm{Lie}\ C_G(A)$.
- $\mathrm{Ad}(a)(x_\alpha) = \alpha(a)\, x_\alpha$

$\Rightarrow (a x_\alpha a^{-1})^t = \alpha(a)\, x_\alpha^t$

$\Rightarrow a^{-1} x_\alpha^t a = \alpha(a)\, x_\alpha^t$

So $\varphi \in \Phi \Rightarrow -\varphi \in \Phi$.

- $N_G(A) \curvearrowright \mathrm{Hom}(A, \mathbb{R}^+)$: $(n \cdot \chi)(a) := \chi(n^{-1} a n)$

It factors through $W := N_G(A)/C_G(A)$, which is called the Weyl group of G.

- $\varphi \in \Phi$, $\omega = [n] \in W \overset{?}{\Rightarrow} \omega \cdot \varphi \in \Phi$

$$\mathrm{Ad}(a)(\underline{\mathrm{Ad}(n)(x_\alpha)}) = \mathrm{Ad}(n)\ \mathrm{Ad}(n^{-1} a n)(x_\alpha)$$
$$= \mathrm{Ad}(n)\left(\alpha(n^{-1} a n)\, x_\alpha \right) = ([n] \cdot \alpha)(a)\ \underline{\mathrm{Ad}(n)(x_\alpha)}$$

$\Rightarrow \mathrm{Ad}(n)\, \mathfrak{g}_\alpha \subseteq \mathfrak{g}_{[n] \cdot \alpha}$, which implies the claim.

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- Let $\mathfrak{a} := \mathrm{Lie}(A)$. Then (Restriction of the Killing form)

$$\mathrm{Tr}\,(\mathrm{ad}(x_1)\,\mathrm{ad}(x_2)) = \sum_{\varphi \in \Phi} \dim \mathfrak{g}_\varphi \ln(\varphi(e^{x_1})) \cdot \ln(\varphi(e^{x_2}))$$

is a positive-definite form on $\mathfrak{a}$ which is $\underline{\underline{W}}$-invariant.

- We can view $\mathrm{Hom}(A, \mathbb{R}^+)$ as the dual space of $\mathfrak{a}$ and identify it with $\mathfrak{a}$ using the above non-degenerate form.

- For any φ, $\exists\, \sigma_\varphi \in W$ s.t. σ_φ is the orthogonal reflection w.r.t. φ (using the above scalar product.); that means

$$\sigma_\varphi(v) = v - \frac{v \cdot \varphi}{\varphi \cdot \varphi}\varphi .$$

- Using this one can get the usual classification of possible Φ's and $\langle \varphi_1, \varphi_2 \rangle := \dfrac{\varphi_1 \cdot \varphi_2}{\varphi_2 \cdot \varphi_2}$.

  In particular, we get

  - There is a set of <u>simple roots</u>; that means $\exists\ \Delta \subseteq \Phi$, $\forall \varphi \in \Phi$, $\exists!\, n_\alpha \in \mathbb{Z}^{\geq 0}$ s.t.

  $$\text{either } \varphi = \sum_{\alpha \in \Delta} n_\alpha \alpha \quad \text{or} \quad \varphi = -\sum_{\alpha \in \Delta} n_\alpha \alpha .$$

  - W acts simply transitively on the collection of sets of simple roots.

  - $A \longrightarrow (\mathbb{R}^+)^{|\Delta|}$, $a \longmapsto (\alpha(a))_{\alpha \in \Delta}$ is an isomorphism.

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- $\forall d \in C_G(A)$, $x_\alpha \in \mathfrak{g}_\alpha$, $a \in A$, $\quad \mathrm{Ad}(a)(\mathrm{Ad}(d)(x_\alpha))$

$$= \mathrm{Ad}(ad)(x_\alpha)$$
$$= \mathrm{Ad}(d)(\mathrm{Ad}(a)\, x_\alpha)$$
$$= \alpha(a)\, \mathrm{Ad}(d)(x_\alpha) \Rightarrow \mathrm{Ad}(d)(x_\alpha) \in \mathfrak{g}_\alpha$$

$\Rightarrow \mathfrak{g}_\alpha$ is $C_G(A)$-invariant.

- For a set of simple roots Δ_0, let

$$^{\triangleleft}A := \{a \in A \mid \alpha(a) > 1 \quad \forall \alpha \in \Delta_0\}.$$

$^{\triangleleft}A$ is called the positive Weyl chamber w.r.t. Δ_0.

- $W \curvearrowright A$ and $\quad A \setminus \bigsqcup_{\omega \in W} \omega \cdot {}^{\triangleleft}A = \bigcup_{\varphi \in \Phi} \ker(\varphi)$.

- $a \in A$ is $\mathbb{R}$-regular, i.e. $C_G(a) = C_G(A) \Rightarrow \exists\, \omega \in W$ s.t.

$$\omega \cdot a \in {}^{\triangleleft}A.$$

- If $|\Delta_0| = r$, then $^{\triangleleft}A$ has $\binom{r}{i}$ faces of dim. i.
- $\omega \in W$, $a \in \overline{^{\triangleleft}A}$, $\omega \cdot a \in \overline{^{\triangleleft}A} \Rightarrow \omega \cdot a = a$.

- For a face $^{\triangleleft}A$ of a Weyl chamber, $\sum_{\alpha(^{\triangleleft}A) > 0} \mathfrak{g}_\alpha$ is a nilpotent Lie subalgebra of $\mathfrak{g}$, which is invariant under $C_G(^{\triangleleft}A)$.

Let $U(^{\triangleleft}A)$ be the corresponding unipotent subgroup, and let

$P(^{\triangleleft}A) := C_G(^{\triangleleft}A)\, U(^{\triangleleft}A)$. It is called a parabolic subgroup.

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. For any $\varphi \in \Phi$, we have already seen that $\mathfrak{g}_\alpha^t = \mathfrak{g}_{-\alpha}$.

Since for any $x \in \mathfrak{g}$, $x - x^t \in \mathrm{Lie}(K)$, we get that

$\mathrm{Lie}(K) + \mathrm{Lie}(P(\triangleleft A)) = \mathfrak{g}$. So $K.P(\triangleleft A)$ is an open

subset of $G \Rightarrow P(\triangleleft A)K/K$ is both open and closed subset

of X. So we get $P(\triangleleft A)K = G$. In particular,

$$G/P(\triangleleft A) \text{ is compact.}$$

- $\triangleleft B_1$ and $\triangleleft B_2$ are two faces of possibly two different Weyl chambers;

$P(\triangleleft B_1) \subseteq P(\triangleleft B_2) \iff \triangleleft B_2$ is a face of $\triangleleft B_1$.

In particular, $P(\triangleleft B)$ is a minimal parabolic $\iff \triangleleft B$ is a Weyl chamber.

. If F is a maximal flat in X, then $F = A \cdot x$ for some maximal polar subgp A and $x \in X$. Then $\triangleleft F = \triangleleft A \cdot x$ is called a Weyl chamber in X.

Lecture 15: Metric definition of maximal boundary

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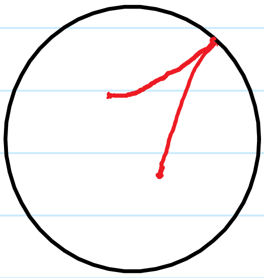
<u>Def.</u> Hausdorff distance of two closed subsets of a metric space is
$$hd(A,B) := \inf \left\{ r \in \mathbb{R}^+ \cup \{\infty\} \;\middle|\; \begin{array}{l} A \subseteq N_r(B) \\ B \subseteq N_r(A) \end{array} \right\}.$$

<u>Def.</u> The maximal boundary X_0 of X is defined as
$$\{ {}^{\triangleleft}F \mid {}^{\triangleleft}F \text{ is a positive Weyl chamber} \} / \sim$$
where ${}^{\triangleleft}F_1 \sim {}^{\triangleleft}F_2 \iff hd({}^{\triangleleft}F_1, {}^{\triangleleft}F_2) < \infty$.

<u>Ex.</u> · <u>Ends of a tree</u>: all the rays $/ \sim$

$$r_1 \sim r_2 \text{ if } |r_1 \triangle r_2| < \infty.$$

· <u>Hyperbolic disc.</u>



$hd(r_1, r_2) < \infty$

$\Updownarrow$

they meet at the same point at the boundary.

<u>Proposition.</u> There is a G-equivariant bijection between X_0 and G/P_0 where P_0 is a minimal parabolic.

<u>Pf.</u> We have $G = P({}^{\triangleleft}A_0)K$ and $P({}^{\triangleleft}A_0) = C_G({}^{\triangleleft}A_0)\,U({}^{\triangleleft}A_0)$

$\forall u \in U({}^{\triangleleft}A_0)$, $\{a^{-1}ua \mid a \in {}^{\triangleleft}A_0\}$ is bounded.