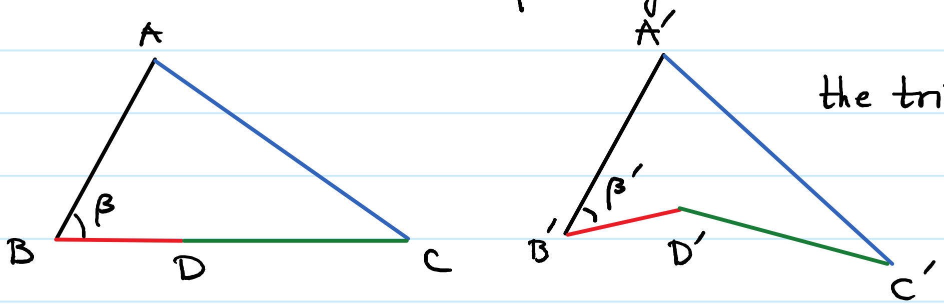
Lecture 14: P(n) is CAT(0)

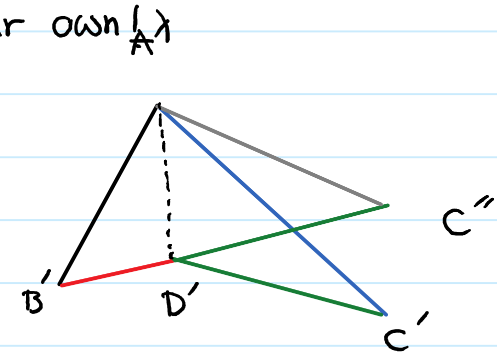
Wednesday, February 22, 2017 9:03 AM

Proposition Let ABC be a triangle in $P(n)$ and $A'B'C'$ be its Euclidean twin. Let $M \in [B,C]$ and $M' \in [B',C']$ s.t. $d(B,M) = d(B',M')$. Then $d(A,M) \leq d(A',M')$.

Lemma (In Euclidean geometry!) In the following picture segments with the same color have equal lengths. Moreover D' is inside the triangle $A'B'C'$ and B, D, and C are collinear. Then $AD \geq A'D'$.



Pf. (It is a cute junior high geometry problem. Try to show it on your own!)



Let's extend $B'D'$ to reach to C'' s.t. $D'C'' = D'C'$. In triangles $A'D'C'$ and $A'D'C''$, we have

$$\left.\begin{array}{l} D'C' = D'C'' \\ A'D' = A'D' \\ \angle A'D'C'' \leq \angle A'D'C' \end{array}\right\} \Rightarrow A'C'' \leq A'C'.$$

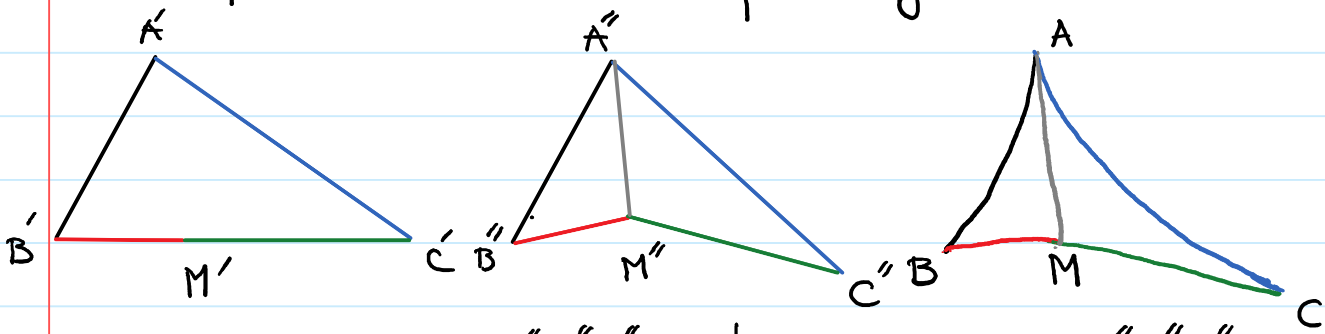
In triangles $A'B'C''$ and ABC, we have $\left.\begin{array}{l} AB = A'B' \\ BC = B'C'' \\ A'C'' \leq A'C' = AC \end{array}\right\} \Rightarrow \beta \geq \beta'.$

In triangles ABD and $A'B'D'$, we have $\left.\begin{array}{l} AB = A'B' \\ BD = B'D' \\ \beta \geq \beta' \end{array}\right\} \Rightarrow AD \geq A'D'.$ ■

Lecture 14: P(n) is a CAT(0) space

Wednesday, February 22, 2017 11:53 AM

Proof of proposition. Let $A''B''M''$ and $B''M''C''$ be the Euclidean twins of ABM and BMC, respectively.



So $\angle BMA \le \angle B''M''A''$ and $\angle CMA \le \angle C''M''A''$.

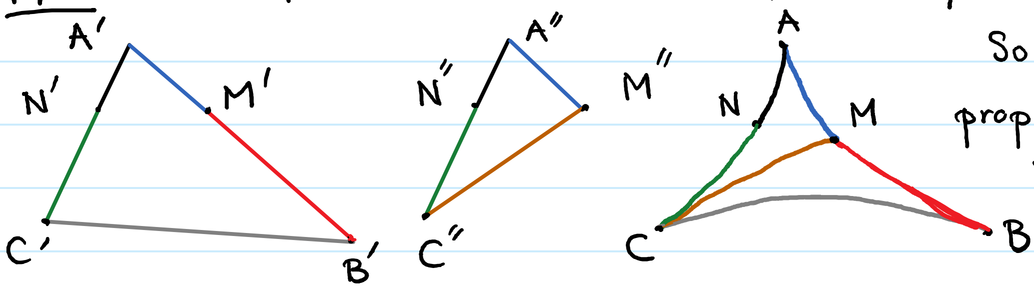
Thus M'' is a point inside $\triangle A''B''C''$. Hence by the previous lemma, $A'M' \ge A''M'' = d(A,M)$. ■

Corollary. Let $A'B'C'$ be the Euclidean twin of ABC. Let $M \in [A,B]$, $M' \in [A',B']$, $N \in [A,C]$, $N' \in [A',C']$ s.t.

$$A'M' = d(A,M), \quad A'N' = d(A,N). \text{ Then}$$

$$d(M,N) \le M'N'.$$

Pf. Let $A''C''M''$ be the Euclidean twin of ABM.



So by the previous proposition, $d(M,N) \le M''N''$. And $d(M,C) = M''C'' \le M'C'$.

In triangles $A'M'C'$ and $A''M''C''$, we have

$$\left.\begin{array}{l} A'M' = A''M'' \\ A'C' = A''C'' \\ M'C' \ge M''C'' \end{array}\right\} \Rightarrow \angle A' \ge \angle A''.$$

In $\triangle A'M'N'$ and $\triangle A''M''N''$,

$$\left.\begin{array}{l} A'M' = A''M'' \\ A'N' = A''N'' \\ \angle A' \ge \angle A'' \end{array}\right\} \Rightarrow$$

$$M'N' \ge M''N'' \ge d(M,N). \quad ■$$

Lecture 14: Energy and center of mass

Thursday, February 23, 2017 8:26 AM

<u>Def.</u> Let F be a compact subset of $P(n)$. We define the energy of a point x_0 w.r.t. F as follows:

$$E_F(x_0) := \int_F d(x_0,x)^2 \, d\mu(x)$$

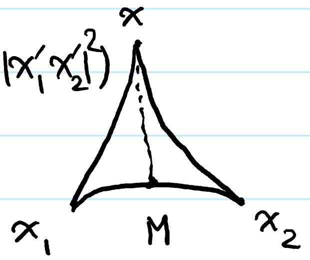
where μ is the volume form induced by the Reimannian metric.

<u>Lemma</u>. For a compact set $F \subseteq X$, there is a unique point $x_F \in X$ which minimizes $E_F(x)$ if $vol(F) \neq 0$.

<u>Pf.</u> By continu. of $d(x_0,x)$, we can show the existance of a point which gives us the minimum.

Now suppose x_1 and x_2 give us the minimum. Let M be the midpoint of $[x_1,x_2]$. Then for any $x \in P(n)$ we have

$$d(x,M)^2 \leq |x'M'|^2 = \frac{1}{4}\left(2|x'x_1'|^2 + 2|x'x_2'|^2 - |x_1'x_2'|^2\right)$$



where $\triangle x'x_1'x_2'$ is the Euclidean twin of $\triangle x x_1 x_2$, and M' is the midpoint of $x_1'x_2'$.

$$\Rightarrow d(x,M)^2 \leq \frac{1}{4}\left(2\,d(x,x_1)^2 + 2\,d(x,x_2)^2 - d(x_1,x_2)^2\right)$$

$$\Rightarrow E_F(M) \leq \frac{1}{4}\left(2\,E_F(x_1) + 2\,E_F(x_2)\right) - vol(F)\, d(x_1,x_2)^2$$

$$\leq E_F(x_1) - vol(F)\, d(x_1,x_2)^2$$ which is a contra. ■

Lecture 14: Maximal compact subgroups are conjugate

Thursday, February 23, 2017 8:42 AM

Def. x_F in the previous lemma is called the center of mass of F.

Proposition. ① Any compact subgroup C of G fixes a point in X.

② Any maximal compact subgroup of G is a conjugate of K.

Pf. ① Let $F := \overline{N_1(C\cdot x_0)}$ be the 1-nbhd of the C-orbit of a point $x_0 \in X$.

$\Rightarrow$ $x \in F$, $\exists\, c' \in C$, $d(x, c'\cdot x_0) \leq 1$

$\Rightarrow$ $\forall c \in C$, $d(c\cdot x, cc'\cdot x_0) \leq 1$

$\Rightarrow$ $c\cdot x \in N_1(C\cdot x_0) = F$. So F is C-invariant.

$$\forall c \in C, \quad E_F(c\cdot x_F) = \int_F d(x, c\cdot x_F)^2\, d\mu(x)$$

$$= \int_F d(c^{-1}\cdot x, x_F)^2\, d\mu(x)$$

$$= \int_F d(x, x_F)^2\, d\mu(x) = E_F(x_F)$$

$\Rightarrow$ $c\cdot x_F = x_F$ because of uniqueness of x_F.

② Let C be a maximal compact subgroup. Then $\exists\, x_0 \in X$ s.t.
$C \subseteq \mathrm{Stab}(x_0) = g_0\, \mathrm{Stab}(I)\, g_0^{-1} = g_0 K g_0^{-1}$ where $g_0 \cdot I = x_0$
By maximality of C, we get $C = g_0 K g_0^{-1}$. ∎

Lecture 14: Lines in a nhbd of a geodesic subspace

Tuesday, February 21, 2017 9:21 AM

Lemma. Suppose a geodesic line L is in $N_d(F)$ where $d \in \mathbb{R}^+$ and F is a geodesic subspace. Then

① $\forall p \in L$, $d(p, F) = d(L, F)$.

② $p_1 \pi(p_1) \pi(p_2) p_2$ is a rectangle, i.e. all the angles are $\pi/2$
$d(p_1, p_2) = d(\pi(p_1), \pi(p_2))$ and $d(p_1, \pi(p_1)) = d(p_2, \pi(p_2))$.

Pf. • Let $s \longmapsto p(s)$ be an arc-length parametrization of L.

Then $s \longmapsto d(p(s), F)$ is a bounded convex function on $\mathbb{R}$.

So it is constant (?).

• We have already said $[p_i, \pi(p_i)] \perp F$.

• Since $d(p, F) = d(p_i, \pi(p_i))$ for any $p \in L$, we get

$$d(\pi(p_i), p_i) = d(\pi(p_i), L).$$

$\Rightarrow$ $\pi_L(\pi(p_i)) = p_i$, which implies $[p_i, \pi(p_i)] \perp L$. ∎

Def. A geodesic subspace F of $\mathbb{P}(n)$ is called flat if the sum of angles of every triangle in F is π.

Lemma. ABC a (non-deg.) triangle s.t. $\alpha + \beta + \gamma = \pi \Rightarrow$
$\exists$ a flat F which contains ABC.

Lecture 14: Flats passing through a triangle

Tuesday, February 21, 2017 11:57 AM

Pf. Suppose $A = I$. Then we have proved that $BC = CB$.

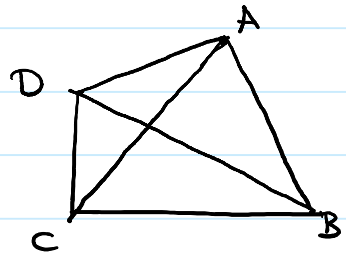
Let $x = \log B$ and $y = \log C$. Now consider

$$F := \{ \exp(t_1 x + t_2 y) \mid t_1, t_2 \in \mathbb{R} \}.$$

Since x and y commute, the Riemannian metric on F is the same as the Euclidean metric on $\mathbb{R}^2$ via the log. map. And so F is flat. ∎

Corollary. Suppose sum of the angles of a quadrilateral is 2π. Then there is a flat which passes through its vertices.

Pf. There are flats F_1 and F_2 which pass through ABC and ACD, respectively. Similarly there is a flat F_3 which passes through BAD. Suppose $A = I$. So taking log does NOT change angle. Since $C\hat{A}D + C\hat{A}B = D\hat{A}B$, all these points are in the same flat. ∎



Lecture 14: Flats in X

Tuesday, February 21, 2017 12:15 PM

<u>Lemma</u>. Flat subspace of G/K which are passing through x_K are $A\cdot x_K$ where A is an analytic abelian subgroup of $\mathbb{P} := \mathbb{P}(n) \cap G$.

<u>Pf.</u> Consider $\log F := \{\log g \mid g \in F\}$. Then it is a commutative set. And so the Euclidean metric on this set is the same as the Rieman. metric on F. Hence $\log F$ is a subspace of $S(n)$ consisting of commu. elements. So $A = \exp(\log F)$ is an analytic abelian subgp of $\mathbb{P}$. The inverse is similar. $\square$

<u>Lemma</u>. Suppose A is a maximal abelian group which is a subset of $\mathbb{P}$. Then A is a maximal polar subgroup of G.

<u>Pf.</u> $d \in C_G(A) \Rightarrow \underset{\forall a \in A}{da = ad} \Rightarrow a^t d^t = d^t a^t \Rightarrow d^t \in C_G(A)$.

$\Rightarrow d = \underbrace{(dd^t)^{1/2}}_{\mathbb{P} \cap C_G(A)} \underbrace{(dd^t)^{-1/2} d}_{K \cap C_G(A)}$. $\Rightarrow d = ak$ for some $a \in A$ and $k \in C_K(A)$.

By maximality of $A \subseteq \mathbb{P} \Rightarrow \mathbb{P} \cap C_G(A) = A$.

So $\mathrm{pol}(d) = a$, which implies that A is a maximal polar subgp. $\blacksquare$