Lecture 13: Flats and sum of the angles

Thursday, February 16, 2017 10:13 AM

Since the equalities hold, we get

① $\log p$ is the Euclidean segment connecting $\log B$ to $\log C$.

② p and $\dot{p}$ commute.

So $\exists f : [0,1] \to [0,1]$ increasing and differ. s.t.

$y(t) = \log p(t) = f(t) \log C + (1 - f(t)) \log B$. And y commutes with $\dot{y}$.

So $f'(t)(\log C - \log B)$ commutes with $f(t)(\log C - \log B) + \log B$

$\Rightarrow f'(t)(\log C - \log B)$ commutes with $\log B$

$\Rightarrow B$ and C commute. ■

Proposition ① For $A, B, C \in X$, let α, β, γ be the angles in the triangle ABC. Then $\alpha + \beta + \gamma \leq \pi$.

And, if $A = I$, then $\alpha + \beta + \gamma = \pi \iff B$ and C commute.

② $a \geq c \sin \alpha$

③ $\gamma \geq \frac{\pi}{2} \Rightarrow b \leq c \cos \alpha$.

Pf. Let A_1, B_1, C_1 be the triangle in the Euclidean plane with sides equal to a, b, c. Then

$$\cos \alpha_1 = \frac{b^2 + c^2 - a^2}{2bc} \leq \cos \alpha \Rightarrow \alpha_1 \geq \alpha$$

Similarly, $\beta_1 \geq \beta$ and $\gamma_1 \geq \gamma$. So $\alpha + \beta + \gamma \leq \alpha_1 + \beta_1 + \gamma_1 = \pi$.

If equality holds, then $\alpha = \alpha_1 \Rightarrow a^2 = b^2 + c^2 - 2bc \cos \alpha$ ✓.

Lecture 13: Thin triangles; diverging rays

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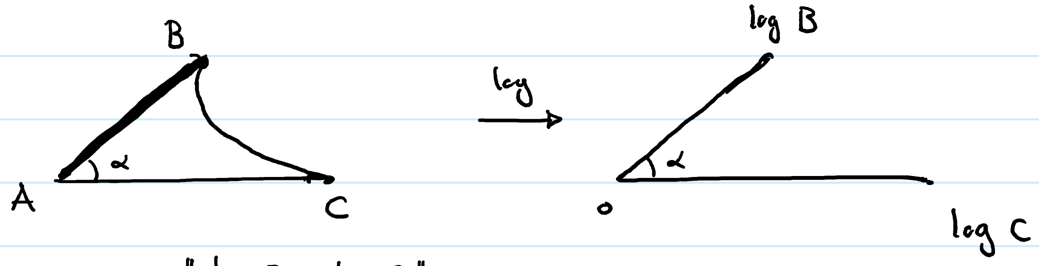
② Suppose $A = I$. Then

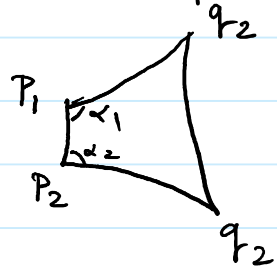


$$\frac{a}{\sin\alpha} \geq \frac{\|\log B - \log C\|}{\sin\alpha} = 2R \geq \|\log C\| = c.$$

③ $$c^2\cos^2\alpha = c^2 - c^2\sin^2\alpha \geq a^2 + b^2 - 2ab\cos\gamma - c^2\sin^2\alpha$$

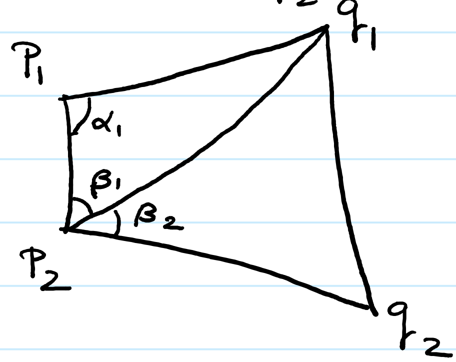
$$\geq a^2 + b^2 - a^2 = b^2. \qquad \blacksquare$$

<u>Lemma</u>. In the following picture $p_1 p_2 q_2 q_1$ is a quadrilateral



Suppose $\alpha_1, \alpha_2 \geq \frac{\pi}{2}$.

Then $d(q_1, q_2) \geq d(p_1, p_2)$.



<u>Pf</u>. Let $a = d(p_1, p_2)$,

$b = d(p_2, q_1)$, and

$c = d(q_1, q_2)$.

Since $\alpha_1 \geq \pi/2$, $b\cos\beta_1 \geq a$. (part ③)

Part ② implies $b\sin\beta_2 \leq c$. So

$$a \leq b\cos\beta_1 = b\sin\left(\frac{\pi}{2} - \beta_1\right) \underset{\boxed{\frac{\pi}{2} - \beta_1 \leq \beta_2}}{\leq} b\sin\beta_2 \leq c. \qquad \blacksquare$$

Lecture 13: CAT(0) space

Tuesday, February 21, 2017 10:47 AM

<u>Proposition.</u> Let ABC be a triangle in $P(n)$ and $A'B'C'$ be its twin in the Eulidean plane, i.e. edges of ABC and $A'B'C'$ have the same length. Let $M \in [B,C]$ and $M' \in [B',C']$ s.t. $d(C,M) = d(C',M')$; and $N \in [A,C]$, $N' \in [A',C']$ s.t. $d(C,N) = d(C',N')$.

Then $d(M,N) \leq d(M',N')$.

<u>Pf.</u> is postponed to the next lecture. (My original argument was flawed!)

<u>Corollary.</u> Suppose $p_t \in [A,B]$ and $q_t \in [A,C]$ s.t.

$$d(A,p_t) = t\, d(A,B) \text{ and } d(A,q_t) = t\, d(A,C) \text{ for } 0 \leq t \leq 1.$$

Then $d(p_t, q_t) \leq t\, d(B,C)$.

<u>Pf.</u> Let $A'B'C'$ be the Euclidean twin of ABC and p_t' and q_t' be

Lecture 13: Convexity of certain function

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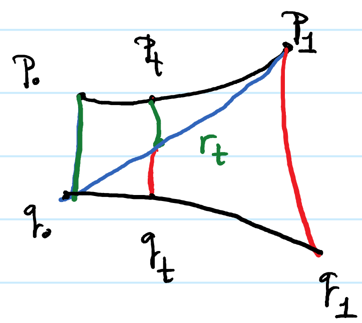
corresponding points in $A'B'C'$. Then by the previous Proposition we have $d(p_t, q_t) \le d(p_t', q_t') = t\, d(B', C') = t\, d(B, C)$.

$$\boxed{\text{from Euclidean geometry}}$$

■

<u>Corollary</u>. Let p_t and q_t be parametrization of geodesic segments $[p_0, p_1]$ and $[q_0, q_1]$ s.t. $d(p_0, p_t) = t\, d(p_0, p_1)$ and $d(q_0, q_t) = t\, d(q_0, q_1)$. Then

$$d(p_t, q_t) \le t\, d(p_1, q_1) + (1-t)\, d(p_0, q_0).$$



<u>Pf.</u> $d(p_t, r_t) \le t\, d(p_0, q_0)$

and $d(r_t, q_t) \le (1-t)\, d(p_1, q_1)$

$\Rightarrow d(p_t, q_t) \le d(p_t, r_t) + d(r_t, q_t)$

$$\le t\, d(p_0, q_0) + (1-t)\, d(p_1, q_1).$$

■

This will be instrumental in the proof of convexity of the function of distance from a convex set.

Lecture 13: Nhbd of a convex set

Tuesday, February 14, 2017 11:38 AM

<u>Proposition</u> Let C be a convex subset of $\mathcal{P}(n)$. Then

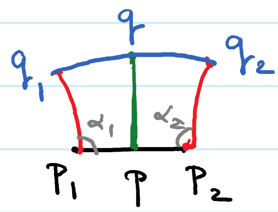
$$N_d(C) := \{x \in \mathcal{P}(n) \mid d(x, C) < d\} \text{ is convex.}$$

<u>Pf.</u> Since $N_d(C) = N_d(\overline{C})$, we can and will assume C is closed.

Let $q_1, q_2 \in N_d(C)$. Since C is closed, $\overline{N_d(q_1)} \cap C$ is compact.

So $\exists p_i \in C$ s.t. $d(q_i, p_i) = d(q_i, C)$. If we show the geodesic segment $[q_1, q_2]$ is a subset of $N_d([p_1, p_2])$, we are done.

Let $q \in [q_1, q_2]$ be such that

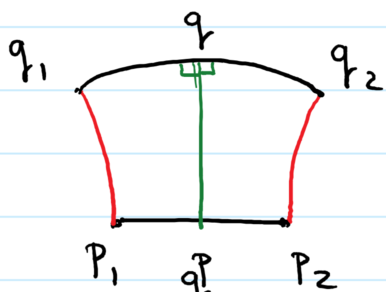


$$d(q, [p_1, p_2]) = \sup_{q' \in [q_1, q_2]} d(q', [p_1, p_2])$$

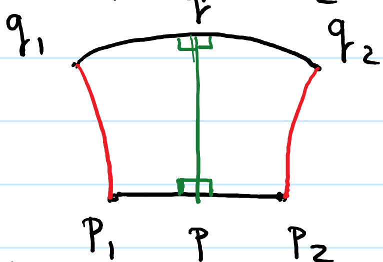
And $p \in [p_1, p_2]$ be such that, $d(q, p) = d(q, [p_1, p_2])$.

① If $q = q_i$, we are done as $d(q_i, p_i) \leq d$.

② If $q \neq q_i$, then



as otherwise we can increase $d(q, [p_1, p_2])$.



②ⓐ If $p \neq p_i$, then $\rightsquigarrow$

Hence $d(p_i, q_i) \geq d(p, q)$, which is a contradiction.

Lecture 13: Nbhd of a convex set

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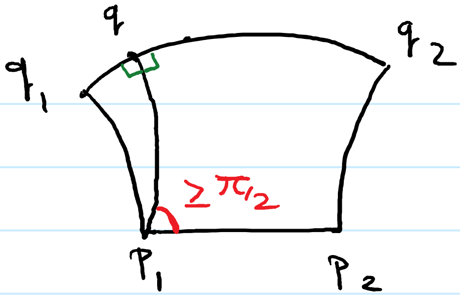


② $\boxed{b}$ $p = p_1$ or p_2.

Since $d(q, p_1)$

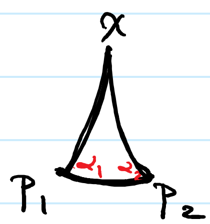
$$= \min d(q_1, [p_1, p_2]) \Rightarrow q\hat{p_1}p_2 \geq \pi/2.$$

$\Rightarrow$ $d(q_2, p_2) \geq d(q, p_1)$ which is a contradiction. ∎

<u>Lemma</u> For any convex closed set C and $x \in P(n)$, there is a unique point $\pi_C(x)$ in C s.t.

$$d(x, \pi_C(x)) = d(x, C).$$

<u>Pf.</u> Suppose $p_1, p_2 \in C$ s.t. $d(x, p_i) = d(x, C)$.



Since $d(x, p_i) = d(x, [p_1, p_2])$,

$\alpha_1, \alpha_2 \geq \pi/2$.

In a triangle sum of the angles are $\leq \pi$. So $p_1 = p_2$. ∎

<u>Def.</u> We say $\pi_C(x)$ is the projection of x to C.

<u>Remark.</u> Let $F \subseteq P(n)$ be a geodesic subspace. Then for any $x \in P(n) \setminus F$, $[x, \pi_F(x)] \perp F$. So, in this case $\pi_F(x)$ is called the orthogonal projection of x onto F. And we have $d(x_1, x_2) \geq d(\pi_F(x_1), \pi_F(x_2))$.

Lecture 13: Convexity of the function of distance from a convex set

Tuesday, February 21, 2017 8:51 AM

Proposition Let $s \mapsto p(s)$ be the arc-length parametrization of a geodesic L in $P(n)$. Let C be a convex set. Then $s \mapsto f_C(s) := d(p(s), C)$ is a convex function, i.e.

$$f_C((1-t)a + tb) \leq (1-t) f_C(a) + t f_C(b).$$

Pf. As $d(x, C) = d(x, \overline{C})$, we can and will assume C is closed.

Let $p_t := p(a(1-t) + tb)$.

$$f_C((1-t)a + tb) = d(p_t, C) \leq d(p_t, [\pi_C(p_0), \pi_C(p_1)])$$

$$:= f_{[\pi_C(p_0), \pi_C(p_1)]}((1-t)a + tb)$$

and $f_C(a) = d(p_0, \pi_C(p_0)) = d(p_0, [\pi_C(p_0), \pi_C(p_1)])$

$$= f_{[\pi_C(p_0), \pi_C(p_1)]}(a),$$

and similarly $f_C(b) = f_{[\pi_C(p_0), \pi_C(p_1)]}(b)$. Let $q_0 = \pi_C(p_0)$, $q_1 = \pi_C(p_1)$

So it is enough to prove the proposition for $C = [q_0, q_1]$.

Hence, for any $t \in [0,1]$, it is enough to find a point q_t in $[q_0, q_1]$ s.t. $d(p_t, q_t) \leq t\, d(p_0, q_0) + (1-t)\, d(p_1, q_1)$. But this we have already done. Let q_t be s.t. $d(q_0, q_t) = t\, d(q_0, q_1)$. ∎