Lecture 11, 12: Geodesics in X

Tuesday, February 14, 2017 9:17 AM

Pf. Let $y(t) := \log p(t) \in S(n)$ (symmetric matrices).

We have to show $\mathrm{Tr}((\dot{y})^2) \le \mathrm{Tr}\left(\left(e^{-y}\left(\frac{d}{dt}e^{y}\right)\right)^2\right)$

$$\frac{d}{dt}e^{y} = \lim_{\Delta t\to 0}\frac{e^{y(t+\Delta t)}-e^{y(t)}}{\Delta t} = \lim_{\Delta t\to 0}\frac{e^{y+\Delta t\,\dot{y}}-e^{y}}{\Delta t}.$$

$$\boxed{\begin{aligned} &y(t+\Delta t) = y+\Delta t\,\dot{y}+u \quad \text{where } \frac{u}{\Delta t}\to 0 \text{ as } \Delta t\to 0.\\ &\|e^{A+B}-e^{A}\| = \left\|\sum_{k=0}^{\infty}\frac{(A+B)^k-A^k}{k!}\right\| \le \sum_{k=0}^{\infty}\frac{(\|A\|+\|B\|)^k-\|A\|^k}{k!} = e^{\|A\|+\|B\|}-e^{\|A\|}\\ &= e^{\|A\|}(e^{\|B\|}-1)\end{aligned}}$$

$$\boxed{e^{y(t+\Delta t)} = e^{y+\Delta t\,\dot{y}}+\omega \quad \text{where } \frac{\omega}{\Delta t}\to 0 \text{ as } \Delta t\to 0}$$

$$\lim_{s\to 0}\frac{e^{A+sB}-e^{A}}{s} \underset{\text{abs. conv.}}{=} \sum_{k=0}^{\infty}\frac{1}{k!}\lim_{s\to 0}\frac{(A+sB)^k-A^k}{s}$$

$$= \sum_{k=1}^{\infty}\frac{1}{k!}\sum_{i=0}^{k-1}A^{i}BA^{k-i-1}$$

$$= \left(\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{i=0}^{k-1}L_A^{i}R_A^{k-i-1}\right)(B).$$

$$\Rightarrow (L_A-R_A)\left(\lim_{s\to 0}\frac{e^{A+sB}-e^{A}}{s}\right) = \left(e^{L_A}-e^{R_A}\right)(B).$$

$$\mathrm{Tr}\left(e^{-y}\frac{d}{dt}e^{y}\right)^2 = \mathrm{Tr}\left(\left(e^{-y/2}\left.\frac{d}{ds}\right|_{s=0}\left(e^{y+s\dot{y}}\right)e^{-y/2}\right)^2\right)$$

To understand $e^{-y/2}\left.\frac{d}{ds}\right|_{s=0}e^{y+s\dot{y}}\,e^{-y/2}$ we let $A=y$ and $B=\dot{y}$

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And we get

$$\left(L_y - R_y\right)\left(e^{-y/2}\frac{d}{ds}\Big|_{s=0} e^{y+s\dot{y}}\, e^{-y/2}\right) =$$

$$L_{e^{-y/2}} R_{e^{-y/2}} \left(L_y - R_y\right)\left(\frac{d}{ds}\Big|_{s=0} e^{y+s\dot{y}}\right) =$$

$$L_{e^{-y/2}} R_{e^{-y/2}} \left(e^{L_y} - e^{R_y}\right)(\dot{y}) = \left(e^{L_{y/2}} e^{R_{-y/2}} - e^{L_{-y/2}} e^{R_{y/2}}\right)(\dot{y})$$

$$\boxed{\begin{aligned} L_{e^A}(B) &= e^A B = \sum \frac{A^k B}{k!} = \sum L_{A^k}(B)/k! \\ &= e^{L_A}(B) \end{aligned}}$$

$$= \left(e^{\left(L_{y/2} - R_{y/2}\right)} - e^{-\left(L_{y/2} - R_{y/2}\right)}\right)(\dot{y})$$

Let $C_y := L_y - R_y$. Then we got

$$C_y\left(e^{-y/2}\frac{d}{ds}\Big|_{s=0} e^{y+s\dot{y}}\, e^{-y/2}\right) = \frac{1}{2}\operatorname{Sinh}\left(C_y/2\right)(\dot{y}).$$

Let $\tau_y(\dot{y}) := e^{-y/2}\frac{d}{ds}\Big|_{s=0} e^{y+s\dot{y}}\, e^{-y/2}$. So

we have proved $\quad C_y \circ \tau_y = \frac{1}{2}\operatorname{Sinh}(C_y/2)$.

And we have to show

$$\operatorname{Tr}(\dot{y}^2) \le \operatorname{Tr}\left(\left(\tau_y(\dot{y})\right)^2\right).$$

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Notice that $(A,B) \mapsto \mathrm{Tr}(AB^t)$ is a dot prod. on $M_n(\mathbb{R})$, and for any $y \in S(n)$, $\langle C_y(A), B\rangle = \mathrm{tr}((yA - Ay)B^t)$

$$= \mathrm{tr}(yAB^t) - \mathrm{tr}(AyB^t) = \mathrm{tr}(AB^ty - AyB^t)$$

$$= \mathrm{tr}(A(y^tB - By^t)^t) = \langle A, C_y(B)\rangle.$$

So C_y is self-adjoint w.r.t. $\langle\ ,\ \rangle$. So it is diagonalizable (w.r.t an orthonormal basis). So we can apply $\frac{\sinh \lambda}{\lambda}$ function to the e.v.'s of C_y. This is an analytic function which is 1 at 0.

Notice that, if $\lambda_1, \dots, \lambda_n$ are e.v's of y, then $\lambda_i - \lambda_j$ are e.v.'s of C_y. And $\frac{\sinh \lambda}{\lambda} \geq 1$. So

$$\langle \dot y, \dot y\rangle \leq \left\langle \frac{\sinh C_y}{C_y}(\dot y), \frac{\sinh C_y}{C_y}(\dot y)\right\rangle.$$

And equality holds if $\dot y$ is in the kernel of C_y, i.e. y and $\dot y$ commute. $\Rightarrow \frac{d}{dt} e^y = \lim_{\Delta t \to 0} \frac{e^{y + \Delta t \dot y} - e^y}{\Delta t} = e^y \lim_{\Delta t \to 0} \frac{e^{\Delta t \dot y} - I}{\Delta t}$

$$\Rightarrow \dot p = p \dot y \text{ and } p = e^y \Rightarrow p \text{ and } \dot p \text{ commute.}$$

If p and $\dot p$ commute, then $\dot y = (\log p)^{\cdot} = p^{-1}\dot p$ commutes with $y = \log p$. $\square$

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The above theorem implies:

If we identify $P(n)$ with $S(n)$ via the logarithmic map and use the Euclidean metric d_S, we get a smaller distance:

For any differentiable curve p in $P(n)$ we have

$$\ell_S(p) = \text{Eucli. length of } \log p \text{ in } S(n)$$

$$= \int \operatorname{Tr}\left((\dot{\overline{\log p}})^2\right) dt$$

$$\leq \int \operatorname{Tr}\left((p^{-1}\dot{p})^2\right) dt = \ell(p).$$

So for any differentiable curve p which connects x to y in $P(n)$ we have,

$$\begin{array}{l}\text{Euclidean distance}\\ \text{between}\\ \log x \text{ and } \log y\\ \text{in } S(n)\end{array} \leq \ell_S(p) \leq \ell(p)$$

For any $x_0 \in X$, let $p(t) = x_0^t$ (warning: it is the 1-para. group; it is NOT the transpose of x_0.)

Then $\dot{p}(t) = (\log x_0) p(t) \Rightarrow p(t)^{-1}\dot{p}(t) = \log x_0 \Rightarrow$

$$\ell(p) = \int_0^1 \sqrt{\operatorname{Tr}((\log x_0)^2)}\, dt = \sqrt{\operatorname{Tr}((\log x_0)^2)} = \|\log x_0 - \log I\|.$$

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Theorem. ① The geodesic which connects I to x is

$$t \mapsto x^t.$$

② For any $x, y \in X$, there is a unique geodesic which connects x to y.

Pf. We have already proved the ① part. We can get the uniqueness of a geodesic connecting I to x using the uniqueness of Euclidean geodesics and the discussed inequality.

. To get the ② part, we notice that $G \subseteq \mathrm{Isom}(X)$ acts transitively on X. So $\exists\, g \in \mathrm{Isom}(X)$ s.t. $g \cdot y = I$.

By part ① there is a unique geodesic which connects $g \cdot y = I$ to $g \cdot x$. So there is a unique geodesic which connects y to x. ■

Theorem. Let A, B, C be three points in X, $a = d(B,C)$, $b = d(A,C)$, $c = d(A,B)$, and α is the angle between the geodesic segments CA and AB. Then

$$a^2 \geq b^2 + c^2 - 2bc \cos\alpha.$$

Lecture 12: Thin triangles and flats

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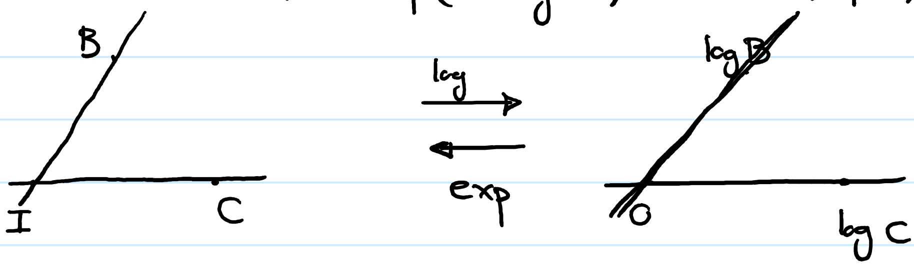
Pf. A Riemannian isometry does NOT change length and angle. And Isom (X) acts transitively on X. So w.l.o.g. we can and will assume $A = I$. So the geodesic segments AB and AC are $t \mapsto \exp(t \log B)$ and $t \mapsto \exp(t \log C)$.



In particular the Euclidean angle between the Euclidean segments $[\log B, 0]$ and $[\log C, 0]$ is α. Hence

$$\|\log B - \log C\|^2 = \|\log B\|^2 + \|\log C\|^2 - 2\|\log B\| \|\log C\| \cos\alpha$$

$$\Rightarrow c^2 + b^2 - 2bc\cos\alpha = \|\log B - \log C\|^2 \leq d(B,C)^2 = a^2.$$

■

Lemma. For points $A = I$, B, C, in the above setting, we get $a^2 = b^2 + c^2 - 2bc\cos\alpha \iff BC = CB$.

Pf. The above equality holds $\iff d(B,C) = \|\log B - \log C\|$.

Let $p(t)$ be the geodesic connecting B to C.

$$\Rightarrow d(B,C) = \int \sqrt{\mathrm{Tr}((p^{-1}\dot{p})^2)}\, dt \geq \int \sqrt{\mathrm{Tr}((\log p)^2)}\, dt \geq \|\log B - \log C\|$$