Lecture 10: Extra property of this deformation in SL(2)

Friday, February 3, 2017 11:40 AM

<u>Claim</u>. Suppose $\lambda = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \Lambda^{(2)}$ has positive distinct eigenvalues and $a_{11} \neq 0$. Then $\rho_t'(\lambda) = g_t\, \lambda^{c_0(t)}\, g_t^{-1}$ where $c_0(t) = c_{\gamma_0}(t)$ as before.

<u>Proof</u>. $\lambda \in \Lambda^{(2)} \Rightarrow \exists\, \gamma \in \Gamma$ s.t. $\gamma = \left[\begin{array}{c|c} \lambda & \\ \hline & I \end{array}\right] \begin{bmatrix} a_1 & & \\ & a_2 & \\ & & \ddots \\ & & & a_n \end{bmatrix}$.

As $a_{11} \neq 0$, for large enough m, $\gamma \gamma_0^m$ is $\mathbb{R}$-regular with positive eigenvalues. So changing a_i's, if needed, we can and will assume γ is $\mathbb{R}$-regular with positive eigenvalues.

- If γ is diagonal, then $\rho_t(\gamma) = \gamma^{c_0(t)}$; and so $\rho_t'(\lambda) = \lambda^{c_0(t)}$.
- If γ is NOT diagonal, then $\exists\, g_t$ s.t.

$$\forall \gamma' \in C_G(\gamma),\quad \rho_t(\gamma') = g_t\, \gamma'^{\,c_\gamma(t)}\, g_t^{-1}.$$

Since $\gamma_1 \in C_G(\gamma_0) \cap C_G(\gamma)$, we have

$$g_t\, \gamma_1^{c_\gamma(t)}\, g_t^{-1} = \gamma_1^{c_0(t)},$$

which implies $c_\gamma(t) = c_0(t)$. And so $\rho_t'(\lambda) = \psi(g_t)\, \lambda^{c_0(t)}\, \psi(g_t)^{-1}$. ∎

Next we want to show $c_0(t)$ is constant (and so $c_0(t) = c_0(0) = 1$.).

The following observation about trace of SL_2 elements is useful:

Lecture 10: Triviality of certain deformations in SL(2)

Saturday, January 28, 2017 10:37 PM

Lemma. Suppose $x, y \in SL_2(F)$. Then

① $\mathrm{tr}(xy) + \mathrm{tr}(xy^{-1}) = \mathrm{tr}(x)\,\mathrm{tr}(y)$

② $\mathrm{tr}(x^2) = \mathrm{tr}(x)^2 - 2$.

Proof ① By Cayley-Hamilton, $y^2 - \mathrm{tr}(y)\, y + I = 0$. So

$$xy + xy^{-1} = \mathrm{tr}(y)\, x, \text{ which implies}$$

$$\mathrm{tr}(xy) + \mathrm{tr}(xy^{-1}) = \mathrm{tr}(y)\,\mathrm{tr}(x).$$

② Let $y = x$. ■

Let $X := \{ \lambda \in \Lambda^{(2)} \mid \lambda = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \mathbb{R}\text{-reg. positive e.v.'s}, \}$.
$a_{11} \neq 0$

Lemma. Suppose $\lambda_1, \lambda_2, \lambda_1^{\pm 1}\lambda_2 \in X$ and $\alpha_1, \alpha_2, \alpha_3 > 1$ are eigenvalues of $\lambda_1, \lambda_2, \lambda_1\lambda_2$, respectively. Then

$$\alpha_3 \in \{ \alpha_1\alpha_2, \alpha_1/\alpha_2, \alpha_2/\alpha_1 \}, \text{ if } c_0(t) \text{ is NOT constant.}$$

Proof. $\mathrm{tr}(\rho_t'(\lambda_1)) = \mathrm{tr}(\lambda_1^{c_0(t)}) = \alpha_1^{c_0(t)} + \alpha_1^{-c_0(t)}$

$\mathrm{tr}(\rho_t'(\lambda_2)) = \mathrm{tr}(\lambda_2^{c_0(t)}) = \alpha_2^{c_0(t)} + \alpha_2^{-c_0(t)}$

$\mathrm{tr}(\rho_t'(\lambda_1\lambda_2)) = \alpha_3^{c_0(t)} + \alpha_3^{-c_0(t)}$ and $\mathrm{tr}(\rho_t'(\lambda_1^{-1}\lambda_2)) = \alpha_4^{c(t)} + \alpha_4^{-c(t)}$

Lecture 10: Finishing the proof

Thursday, January 26, 2017 10:39 PM

$$\mathrm{tr}(\rho_t(\lambda_1)\rho_t(\lambda_2)) + \mathrm{tr}(\rho_t(\lambda_1)^{-1}\rho_t(\lambda_2)) = \mathrm{tr}(\rho_t(\lambda_1))\,\mathrm{tr}(\rho_t(\lambda_2))$$

$$\Rightarrow \alpha_3^{c_0(t)} + \alpha_3^{-c_0(t)} + \alpha_4^{c_0(t)} + \alpha_4^{-c_0(t)} = (\alpha_1^{c_0(t)} + \alpha_1^{-c_0(t)})(\alpha_2^{c_0(t)} + \alpha_2^{-c_0(t)})$$

$$= (\alpha_1\alpha_2)^{c_0(t)} + (\alpha_1\alpha_2)^{-c_0(t)} + (\alpha_1/\alpha_2)^{c_0(t)} + (\alpha_1/\alpha_2)^{-c_0(t)}.$$

For a simply-connected region which contains $\mathbb{R}^+$, we define α_i^z, $(\alpha_i/\alpha_j)^z$. Since $c_0(t)$ is NOT constant, $\{c_0(t)\}$ has a limit point. Since zeros of a non-zero analytic function do not have a limit point, $\forall\, r \in \mathbb{R}^+$, $\alpha_3^r + \alpha_3^{-r} + \alpha_4^r + \alpha_4^{-r} = (\alpha_1\alpha_2)^r + (\alpha_1\alpha_2)^{-r} + (\alpha_1/\alpha_2)^r + (\alpha_1/\alpha_2)^{-r}$.

Hence α_3 is a root of $(x-\alpha_1\alpha_2)(x-(\alpha_1\alpha_2)^{-1})(x-\alpha_1/\alpha_2)(x-\alpha_2/\alpha_1)$. ∎

<u>Lemma.</u> Suppose $c_0(t)$ is NOT constant and $\lambda_1, \lambda_2, \lambda_1^{\pm 1}\lambda_2 \in X$. Then

$$\mathrm{tr}(\lambda_1)^2 + \mathrm{tr}(\lambda_2)^2 + \mathrm{tr}(\lambda_1\lambda_2)^2 - 4 = \mathrm{tr}(\lambda_1)\,\mathrm{tr}(\lambda_2)\,\mathrm{tr}(\lambda_1\lambda_2).$$

<u>Proof</u>. $(x-\alpha_1\alpha_2)(x-\alpha_1/\alpha_2)(x-\alpha_2/\alpha_1)(x-1/\alpha_1\alpha_2) =$

$$\left(x^2 - \left(\alpha_1\alpha_2 + \frac{1}{\alpha_1\alpha_2}\right)x + 1\right)\left(x^2 - (\alpha_1/\alpha_2 + \alpha_2/\alpha_1)x + 1\right) =$$

$$x^4 - \left(\alpha_1\alpha_2 + (\alpha_1\alpha_2)^{-1} + \frac{\alpha_1}{\alpha_2} + \frac{\alpha_2}{\alpha_1}\right)x^3 + \left(2 + \left(\alpha_1^2 + \frac{1}{\alpha_2^2} + \alpha_2^2 + \frac{1}{\alpha_1^2}\right)\right)x^2$$

$$-\left(\alpha_1\alpha_2 + (\alpha_1\alpha_2)^{-1} + \frac{\alpha_1}{\alpha_2} + \frac{\alpha_2}{\alpha_1}\right)x + 1 =$$

$$x^4 - \mathrm{tr}(\lambda_1)\,\mathrm{tr}(\lambda_2)\,x^3 + (2 + \mathrm{tr}(\lambda_1^2) + \mathrm{tr}(\lambda_2^2))\,x^2$$

$$-\mathrm{tr}(\lambda_1)\,\mathrm{tr}(\lambda_2)\,x + 1 =$$

$$x^4 - \mathrm{tr}(\lambda_1)\mathrm{tr}(\lambda_2)x^3 + (\mathrm{tr}(\lambda_1)^2 + \mathrm{tr}(\lambda_2)^2 - 2)x^2 - \mathrm{tr}(\lambda_1)\mathrm{tr}(\lambda_2)x + 1.$$

Lecture 10: Finishing the proof

Tuesday, February 7, 2017 11:53 AM

By the previous lemma, we have

$$(\lambda_1\lambda_2)^4 - tr(\lambda_1)\,tr(\lambda_2)\,(\lambda_1\lambda_2)^3 + \left(tr(\lambda_1)^2 + tr(\lambda_2)^2 - 2\right)(\lambda_1\lambda_2)^2$$
$$- tr(\lambda_1)\,tr(\lambda_2)\,(\lambda_1\lambda_2) + I = 0$$

$$\Rightarrow tr\left((\lambda_1\lambda_2)^2\right) - tr(\lambda_1)\,tr(\lambda_2)\,tr(\lambda_1\lambda_2) + 2\left(tr(\lambda_1)^2 + tr(\lambda_2)^2 - 2\right)$$
$$+ tr\left((\lambda_1\lambda_2)^{-2}\right) - tr(\lambda_1)\,tr(\lambda_2)\,tr\left((\lambda_1\lambda_2)^{-1}\right) = 0$$

$$\Rightarrow tr\left((\lambda_1\lambda_2)^2\right) + tr(\lambda_1)^2 + tr(\lambda_2)^2 - 2 = tr(\lambda_1)\,tr(\lambda_2)\,tr(\lambda_1\lambda_2)$$

$$\Rightarrow tr(\lambda_1\lambda_2)^2 + tr(\lambda_1)^2 + tr(\lambda_2)^2 - 4 = tr(\lambda_1)\,tr(\lambda_2)\,tr(\lambda_1\lambda_2).$$

∎

Notice that $\lambda_0 := \psi(\gamma_0) = \begin{bmatrix} \alpha & \\ & \alpha^{-1} \end{bmatrix} \in X$. And, by Borel density, $\exists$ $\lambda = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Lambda^{(2)}$ s.t. $abcd \neq 0$. Hence

$$\lambda^{-1}\lambda_0^m\lambda = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} \alpha^m & \\ & \alpha^{-m} \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$= \begin{bmatrix} d\alpha^m & -b\alpha^{-m} \\ -c\alpha^m & a\alpha^{-m} \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ad\alpha^m - bc\alpha^{-m} & bd(\alpha^m - \alpha^{-m}) \\ -ac(\alpha^m - \alpha^{-m}) & ad\alpha^{-m} - bc\alpha^m \end{bmatrix}$$

- $ad\alpha^m - bc\alpha^{-m} = bc\alpha^m + \alpha^m - bc\alpha^{-m}$
$$= bc(\alpha^m - \alpha^{-m}) + \alpha^m > \alpha^m > 0$$

- $ad\alpha^{-m} - bc\alpha^m \Rightarrow \dfrac{ad}{bc} = \alpha^{2m}$ which can happen at most for $\underline{1}$ value of m. $\Rightarrow$ all the entries of $\lambda^{-1}\lambda_0^m\lambda$ are non-zero and its $\underline{11}$ entry is arbitrarily large. So, for $n \underset{m_0}{\gg} 1$, $\lambda^{-1}\lambda_0^m\lambda\,\lambda^n$ is

Lecture 10: Finishing the proof

Tuesday, February 7, 2017 12:17 PM

So there is $\lambda_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in X$, $abcd \neq 0$ s.t. $\lambda_1 \lambda_0^{\pm 1} \in X$.

Hence $tr(\lambda_1\lambda_0)^2 + tr(\lambda_1)^2 + tr(\lambda_0)^2 - 4$

$$= tr(\lambda_1\lambda_0)\, tr(\lambda_1)\, tr(\lambda_0)$$

$$\Rightarrow (a\alpha + d\alpha^{-1})^2 + (a+d)^2 + (\alpha+\alpha^{-1})^2 - 4$$

$$= (a\alpha + d\alpha^{-1})(a+d)(\alpha+\alpha^{-1})$$

$$\Rightarrow \cancel{a^2\alpha^2} + 2ad + d^2\alpha^{-2} + \cancel{a^2} + 2ad + d^2 + \alpha^2 + \alpha^{-2} + 2 - 4$$

$$= (a^2\alpha + ad\,\alpha^{-1} + ad\,\alpha + d^2\alpha^{-1})(\alpha+\alpha^{-1})$$

$$= \cancel{a^2\alpha^2} + ad + ad\,\alpha^2 + d^2 + \cancel{a^2} + ad\,\alpha^{-2} + ad + d^2\alpha^{-2}$$

$$\Rightarrow \alpha^2 + \alpha^{-2} - 2 = ad\,(\alpha^2 + \alpha^{-2} - 2)$$

$\Rightarrow$ either $\alpha = \alpha^{-1}$ or $ad = 1$.

$\Rightarrow$ $bc = 0$ which is a contradiction. $\blacksquare$

Lecture 10: Summary of proof of Selberg's local rigidity

Thursday, February 9, 2017 8:46 AM

Theorem. Suppose $\rho_t : \Gamma \to SL_n(\mathbb{R})$ is an algebraic family of injections and $\rho_t(\Gamma)$ is a cocompact lattice in $SL_n(\mathbb{R})$; and $\rho_0(\gamma) = \gamma$. Then $\exists g_t \in SL_n(\mathbb{R})$, $\rho_t(\gamma) = g_t \gamma g_t^{-1}$, for any $\gamma \in \Gamma$.

Step 1. It is enough to prove: $\forall t$, $tr(\rho_t(\gamma)) = tr(\gamma)$.

We used Borel density to show the $\mathbb{R}$-span of $\rho_t(\Gamma)$ is $M_n(\mathbb{R})$; and then used degeneracy of $(x,y) \mapsto tr(xy)$ to extend ρ_t to an algebra isomorphism $\hat{\rho}_t : M_n(\mathbb{R}) \to M_n(\mathbb{R})$; finished the proof using Skolem-Noether.

Step 2. It is enough to prove:

$$\forall \gamma \in \Gamma,\ \mathbb{R}\text{-regular},\ \forall t,\ tr(\rho_t(\gamma)) = tr(\gamma).$$

(i) we proved:

a: $\mathbb{R}$-regular with positive e.v.'s $\Rightarrow \exists$ a nbhd U_G of I s.t. $\forall m \in \mathbb{Z}^+$, $U_G\, a^m\, U_G$ consists of $\mathbb{R}$-reg. elements with positive e.v.'s.

(ii) We used (i) to conclude $\Gamma^{(r)} \neq \emptyset$ where

$\Gamma^{(r)} := \{\gamma \in \Gamma \mid \mathbb{R}\text{-regular with positive e.v.'s}\}$.

Lecture 10: Summary of proof of Selberg's local rigidity

Thursday, February 9, 2017 9:01 AM

(iii) We proved:

For any $\mathbb{R}$-regular element a, there is a proper Zariski closed set S_a s.t.

① $\forall g \in G \setminus S_a$, $n \underset{g}{\gg} 1$, $g a^n$ is $\mathbb{R}$-regular.

② No conjugacy class is a subset of S_a.

(iv) By (ii), we take $\gamma_0 \in \Gamma^{(r)}$. Using Borel density and (iii-2), $\forall \gamma \in \Gamma$, $\exists \gamma' \in \Gamma$ s.t. $\gamma' \gamma \gamma'^{-1} \notin S_{\gamma_0}$.

$\Rightarrow$ $\gamma' \gamma \gamma'^{-1} \gamma_0^m$ is $\mathbb{R}$-regular for $m \gg 1$.

$\Rightarrow$ After diag. γ_0, we can conclude:

$$\operatorname{tr}(\rho_t(\gamma'\gamma\gamma'^{-1})) = \operatorname{tr}(\gamma'\gamma\gamma'^{-1})$$

$$\Rightarrow \operatorname{tr}(\rho_t(\gamma)) = \operatorname{tr}(\gamma).$$

Step 3. For any $\gamma \in \Gamma^{(r)}$, (i) $\rho_t(\gamma) \in \rho_t(\Gamma)^{(r)}$.

(ii) $C_{\rho_t(\Gamma)}(\rho_t(\gamma))$ is a lattice in $C_G(\rho_t(\gamma))$.

(iii) $C_G(\rho_t(\gamma))^\circ$ is $\mathbb{R}$-conjugate of $\{\operatorname{diag}(a_1,\dots,a_n) \mid a_i > 0, \prod a_i = 1\}$.

(iv) $\theta_t : \Delta_\gamma \to \Delta_{\rho_t(\gamma)}$, $\theta_t(v) = c_\gamma(t)\, v$ where

$\Delta_\gamma = \log(C_\Gamma(\gamma) \cap C_G(\gamma)^\circ) \subseteq \operatorname{Lie} C_G(\gamma)$, and

$\theta_t(v) = \log(\rho_t(e^v))$.

Lecture 10: Summary of proof of Selberg's local rigidity

Thursday, February 9, 2017 9:18 AM

(i) $\Gamma \subseteq G$ cocompact lattice $\Rightarrow$ $\forall\, \gamma \in \Gamma$, $C_\Gamma(\gamma) \subseteq C_G(\gamma)$ cocompact lattice.

(ii) (Borel's density) $\left.\begin{array}{l}\Gamma \subseteq G \text{ lattice} \\ H: \text{Zariski-closure of } \Gamma\end{array}\right\} \Rightarrow G^+ \subseteq H$, where G^+ is the subgp gen. by unipotent elements.

(iii) In a cocompact lattice of $SL_n(\mathbb{R})$, any element is semisimple

(iv) If t_0 is the first time that $\rho_t(\gamma_0)$ is NOT $\mathbb{R}$-regular, then $\rho_{t_0}(\gamma_0)$ is diag. and at least two of its e.v.'s are equal $\Rightarrow$ $C_G(\rho_{t_0}(\gamma_0))^+$ contains a copy of $SL_2(\mathbb{R})$.

So $C_{\rho_{t_0}(\Gamma)}(\rho_{t_0}(\gamma_0)) = \rho_{t_0}(C_\Gamma(\gamma_0))$ cannot be abelian.

So $\rho_t(\gamma_0) \in \rho_t(\Gamma)^{(r)}$ and we get parts (ii) and (iii).

To get the last part of this step:

(v) $\theta_t(\Delta_\gamma \cap \mathfrak{a}_\sigma^+) = \Delta_{\rho_t(\gamma)} \cap \mathfrak{a}_\sigma^+$ for any Weyl chamber $\mathfrak{a}_\sigma^+$ and the set of directions in any lattice of $\mathfrak{a}$ is dense in the unit sphere of $\mathfrak{a}$

$\Rightarrow$ $\theta_t(\mathfrak{a}_\sigma^+) = \mathfrak{a}_\sigma^+$ for any Weyl chamber.

(vi) $\left.\begin{array}{l}\theta: \mathfrak{a} \to \mathfrak{a} \text{ linear} \\ \theta(\mathfrak{a}_\sigma^+) = \mathfrak{a}_\sigma^+\end{array}\right\} \Rightarrow \theta(v) = c\, v.$

<u>Step 4.</u> $\forall\, \gamma \in \Gamma^{(r)}$, $c_\gamma(t) = 1$; which implies $tr(\rho_t(\gamma)) = tr(\gamma)$.

Lecture 10: Summary of proof of Selberg's local rigidity

Thursday, February 9, 2017 9:38 AM

(i) Let $\gamma_0 \in \Gamma^{(r)}$; after conjugation we assume

$$\rho_t(\gamma_0) = \mathrm{diag}(\alpha_1(t), \dots, \alpha_n(t)).$$

Either $c_{\gamma_0}(t) = 1$ or $\forall v \in \Delta_0, \exists c \in \mathbb{R}^+$ s.t. $cv \in \Delta_{\gamma_0}$ where $\Delta_0 := \{\mathrm{diag}(x_1, \dots, x_n) \mid x_i \in \mathbb{Z}, \sum x_i = 0\}$.

(ii) If $c_{\gamma_0}(t) \neq 1$, then $\exists \gamma_1 = \mathrm{diag}(a_1, a_1, a_2, \dots, a_{n-1}) \in \Gamma$, where $a_1 > a_2 > \dots > a_{n-1}$.

(iii) Since $\rho_t(\gamma_1) = \mathrm{diag}(a_1^{c_{\gamma_0}(t)}, a_1^{c_{\gamma_0}(t)}, \dots, a_{n-1}^{c_{\gamma_0}(t)})$,

$\rho_t(C_\Gamma(\gamma_1))$ is a lattice in $\tilde{H} := \left\{ \begin{bmatrix} g & \\ & a_3 & \\ & & \ddots & \\ & & & a_n \end{bmatrix} \middle| \det(g)\, a_3 \cdots a_n = 1 \right\}$.

(iv) Let $H = \tilde{H}^0$. Then $H \xrightarrow[\sim]{\phi} SL_2(\mathbb{R}) \times (\mathbb{R}^+)^{n-2}$

$$\begin{bmatrix} g & \\ & a_1 & \\ & & \ddots & \\ & & & a_{n-2} \end{bmatrix} \longmapsto \left(\frac{1}{\sqrt{\det g}}\, g, (a_1, \dots, a_{n-2}) \right),$$

and

$$\begin{bmatrix} g & \\ & a_1 & \\ & & \ddots & \\ & & & a_{n-2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{\det g}}\, g & \\ & I \end{bmatrix} \underbrace{\begin{bmatrix} \sqrt{\det g}\, I_2 & \\ & a_1 & \\ & & \ddots & \\ & & & a_{n-2} \end{bmatrix}}_{Z(H)}.$$

(v) Let $\psi : H \to SL_2(\mathbb{R})$, $\psi = pr_{SL_2(\mathbb{R})} \circ \phi$. Then

- $\Lambda_t^{(2)} := \psi(\rho_t(C_\Gamma(\gamma_1)))$ is a cocompact lattice in $SL_2(\mathbb{R})$
- $\rho_t' : \Lambda_0^{(2)} \longrightarrow SL_2(\mathbb{R})$, $\rho_t'(\psi(\gamma)) := \psi(\rho_t(\gamma))$ is a well-defined deformation.

Lecture 10: Summary of proof of Selberg's local rigidity

Thursday, February 9, 2017 10:24 AM

(vi) For any $\lambda \in (\Lambda_0^{(2)})^{(n)}$, $\mathrm{tr}(\rho_t'(\lambda)) = \mathrm{tr}(\lambda^{c_0(t)})$

where $c_0(t) = c_{\gamma_0}(t)$.

(vii) We use trace identities is SL_2, and our earlier constructions of $\mathbb{R}$-reg. elements in $\Lambda_0^{(2)}$ to get a contradiction.