Lecture 7: Translates of powers of an $\mathbb{R}$-regular element

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Next we will prove the following:

Theorem. Let $a_0 = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ where $\lambda_1 > \cdots > \lambda_n > 0$ and $\prod \lambda_i = 1$.

Then ① for any $g \in N_- \underbrace{\tilde{C}_{SL_n(\mathbb{R})}(A)}^{\text{(all diag.)}} N_+$, for $k \gg_g 1$, $g a_0^k$ is $\mathbb{R}$-regular.

② No conjugacy class is a subset of

$$S_{a_0} := SL_n(\mathbb{R}) \setminus N_- C_{SL_n(\mathbb{R})}(A) N_+ .$$

(Notice that S_{a_0} is a proper Zariski-closed subset of $SL_n(\mathbb{R})$.)

Lemma. Let $C_0 \subseteq N^-$ be a compact subset and $t > 1$. Then there is a compact subset $C_1 \subseteq N^-$ such that

$$C_0 a \subseteq \{ g a g^{-1} \mid g \in C_1 \}$$

for any $a \in A_t$.

Proof. Let $U_k := \left\{ \begin{bmatrix} 1 & & & & \\ 0 & \ddots & & & \\ \vdots & & \ddots & & \\ 0 & & & \ddots & \\ * & \cdots & \underbrace{0 \cdots 0}_{k} & & 1 \end{bmatrix} \in N^- \right\}$. By induction on k,

we show, for a compact subset $C_0^{(k)} \subseteq U_k$, there is a compact subset $C_1^{(k)} \subseteq U_k$ s.t.

$$C_0^{(k)} a\, U_{k+1} \subseteq \{ g a g^{-1} \mid g \in C_1^{(k)} \} U_{k+1} .$$

Notice that U_k is a normal subgroup of $N^- A$.

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For $g=\begin{bmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ \vdots & \ddots & \ddots & & & & & \\ 0 & & \ddots & \ddots & & & & \\ g_{k+1,1} & & & \ddots & \ddots & & & \\ \vdots & \ddots & & & \ddots & \ddots & & \\ g_{n,1} & \cdots & g_{n,n-k} & 0 & \cdots & 0 & 1 \end{bmatrix}$ we have

$$(gag^{-1}a^{-1})_{ij} = g_{ij} - a_i a_j^{-1} g_{ij} = (1-a_i a_j^{-1})\, g_{ij}$$

for $(i,j)\in\{(k+1,1),(k+2,2),\cdots,(n,n-k)\}$.

Since $a\in A_t$, for $i>j$, we have $0<a_i a_j^{-1}<\frac{1}{t}$. Hence, for $i>j$, $1-\frac{1}{t}<1-a_i a_j^{-1}<1$. So for a given c from a bounded subset of U_k, $\exists\, g$ from a bounded subset of N^- s.t.

$$gag^{-1}a^{-1}\, U_{k+1} = c\, U_{k+1}. \qquad \square$$

<u>Proof of theorem</u> $\boxed{1}$ Let $g=n_- d\, n_+$ where $n_-\in N^-$, $n_+\in N^+$, and $d\in A$. So $ga_0^k=(n_- d a_0^k)(a_0^{-k} n_+ a_0^k)$. By the previous lemma, $\exists$ a compact subset C_1 of N_- such that $n_- d a_0^k = u_- d a_0^k u_-^{-1}$ for some $u_-\in C_1$. Now applying Proposition$'$ for $C:=C_1$ and (small) nbhds O_A, O_G of I in A and $SL_n(\mathbb{R})$ we get a nbhd U_G of I such that:

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$$\forall\, u_- \in C_1,\ d' \in A_t,\quad u_- a' u_-^{-1}\, U_G \subseteq u_- \{ g' a' d' g'^{-1} \mid g' \in \mathcal{O}_G,\ d' \in \mathcal{O}_A \} u_-^{-1}.$$

Now, for large enough k, (1) $d a_0^k \in A_t$;

(2) $a_0^{-k} n_+ a_0^k \in U_G$.

Therefore
$$n_- d n_+ a_0^k = (n_- d a_0^k)(a_0^{-k} n_+ a_0^k)$$
$$\in (u_- d a_0^k u_-^{-1})\, U_G \subseteq u_- \{ g' d a_0^k d' g'^{-1} \mid g' \in \mathcal{O}_G,\ d' \in \mathcal{O}_A \} u_-^{-1}.$$

Hence, for large enough k, $g a_0^k$ is $\mathbb{R}$-regular and the j^{th} eigenvalue of $g a_0^k$ is in $\left(\frac{1}{\eta} \lambda_j^k d_j,\ \eta \lambda_j^k d_j\right)$ where $d = \mathrm{diag}(d_1, \ldots, d_n)$.

$\boxed{\text{Part (2)}}$ Suppose $\{ g x g^{-1} \mid g \in SL_n(\mathbb{R}) \} \subseteq S_{a_0}$. (Just using permutation matrices we can get a contradiction, but the following proof works for other semisimple Lie groups.)

Suppose $E \subseteq S_{a_0}$ is Zariski-closed and $g E g^{-1} = E$ for any $g \in SL_n(\mathbb{R})$. So $\mathcal{O} := SL_n(\mathbb{R}) \setminus E \supseteq N_- A N_+$ is G-invar. under conjugation. Hence $N_+ N_- A N_+ \subseteq \bigcup_{n_+ \in N_+} n_+ \mathcal{O} n_+^{-1} = \mathcal{O}$.

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$\Rightarrow \mathcal{O} \supseteq N_+ A N_- \cdot N_- A N_+$ (AN_ is a subgp)

Let $\bar{J} := N_- A N_+$. So $\bar{J}^{-1} \bar{J} \subseteq \mathcal{O}$.

A similar argument gives us $J^{-1} J \subseteq \mathcal{O}$ where

$$J = N_- \, C_{SL_n(\mathbb{R})}(A) \, N_+ .$$

Since J is a Zariski-open, dense subset of $SL_n(\mathbb{R})$,

$\forall g \in SL_n(\mathbb{R})$, $Jg \cap J \neq \emptyset$. So $g \in J^{-1} J \subseteq \mathcal{O}$.

$\Rightarrow \mathcal{O} = SL_n(\mathbb{R})$, which contradicts $E \neq \emptyset$. ■

Lecture 7: Finding $\mathbb{R}$-regular elements in Γ

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Proposition. Let Γ be a lattice in $SL_n(\mathbb{R})$. Then for any $\lambda_1 > \cdots > \lambda_n > 0$, $\lambda_1 \cdots \lambda_n = 1$ and $\eta > 1$, there are positive integers $m_1 < m_2 < \cdots$ and $\gamma_1, \gamma_2, \ldots \in \Gamma$ such that the following hold:

① γ_i's are $\mathbb{R}$-regular with positive eigen-values

$$\lambda_1(\gamma_i) > \lambda_2(\gamma_i) > \cdots > \lambda_n(\gamma_i) > 0.$$

② $$\frac{1}{\eta} \lambda_j^{m_i} < \lambda_j(\gamma_i) < \eta \lambda_j^{m_i}$$

for $1 \leq j \leq n$.

Proof. Let $a_0 := \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. By the corollary of the main proposition, we got a nbhd U_G of I is $SL_n(\mathbb{R})$ such that any element of $U_G a_0^{m_i} U_G$ satisfies ① and ②.

Let $U_G' := U_G \cap U_G^{-1}$ and consider $a_0^i U_G' \Gamma / \Gamma$. Since all of them have the same measure, $\exists\, 0 < k_1 < k_2$ s.t.

$$a_0^{k_1} U_G' \Gamma \cap a_0^{k_2} U_G' \Gamma \neq \emptyset \Rightarrow \Gamma \cap U_G'^{-1} a_0^{k_2 - k_1} U_G' \neq \emptyset$$

Repeating this argument for powers of a_0, we get $\exists\, m_1 < m_2 < \cdots$ and $\gamma_i \in \Gamma \cap U_G' a_0^{m_i} U_G'$ which completes the proof. ∎

Lecture 7: Set of $\mathbb{R}$-regular elements of Γ

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Let $\Gamma^{(r)} := \{\gamma \in \Gamma \mid \gamma \text{ is } \mathbb{R}\text{-regular}\}$. So the previous Proposition implies $\exists\, \gamma_0 \in \Gamma^{(r)}$ with positive eigenvalues. Hence for any $\gamma_1 \in \Gamma \setminus S_{\gamma_0}$ (as in the theorem on translate of powers of $\mathbb{R}$-regular elements) we have

$$\gamma_1 \gamma_0^n \in \Gamma^{(r)} \quad \text{for any } n \gg_{\gamma_1, \gamma_0} 1.$$

Claim. $\exists\, \gamma \in \Gamma$ such that $\gamma \gamma_1 \gamma^{-1} \notin S_{\gamma_0}$.

Pf of Claim If not, the Zariski-closure S of $\{\gamma\gamma_1\gamma^{-1} \mid \gamma \in \Gamma\} \subseteq S_{\gamma_0}$. Since S is Γ-invariant (under conjugation), by Borel's density theorem S is $SL_n(\mathbb{R})$-invariant (under conjugation). This contradicts part ② of Theorem which says no conjugacy class is a subset of S_{γ_0}. $\square$

Overall we get:

Corollary. $\forall\, \gamma_0 \in \Gamma^{(r)}$, $\gamma_1 \in \Gamma$, $\exists\, n_0 \in \mathbb{Z}^+$, $\gamma \in \Gamma$ s.t.

$$\gamma \gamma_1 \gamma^{-1} \gamma_0^n \in \Gamma^{(r)} \quad \text{for } n \geq n_0.$$

Lecture 7: Trace rigidity of -regular elements implies trace rigidity (Step 2)

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Proof of Step 2 let $\gamma_0 \in \Gamma^{(r)}$ with positive eigen-value.

So for small values of t, $\rho_t(\gamma_0)$ is $\mathbb{R}$-regular with positive eigen-values. By the assumption, for any $n \in \mathbb{Z}^+$,

$tr(\gamma_0^n) = tr(\rho_t(\gamma_0)^n)$. Hence γ_0 and $\rho_t(\gamma_0)$ have the same eigen-values. Therefore $\exists\, g_t \in SL_n(\mathbb{R})$ s.t.

① $g_t \to I$, ② $g_t \rho_t(\gamma_0) g_t^{-1} = \gamma_0$. So after changing ρ_t by a (continuous) conjugation, we can assume $\rho_t(\gamma_0) = \gamma_0$.

Further we can assume $\gamma_0 = diag(\lambda_1, \dots, \lambda_n)$ for $\lambda_1 > \dots > \lambda_n > 0$ (Changing Γ by a conjugation.)

For any $\gamma_1 \in \Gamma$, $\exists\, \gamma \in \Gamma$, $n_0 \in \mathbb{Z}^+$ s.t.

$$\forall k \geq n_0, \quad \gamma\gamma_1\gamma^{-1}\gamma_0^k \in \Gamma^{(r)}.$$

Hence
$$tr(\gamma\gamma_1\gamma^{-1}\gamma_0^k) = tr(\rho_t(\gamma\gamma_1\gamma^{-1})\rho_t(\gamma_0)^k)$$
$$= tr(\rho_t(\gamma\gamma_1\gamma^{-1})\,\gamma_0^k). \quad \circledast$$

Notice $tr\left(\begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix}\begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix}\right) = \sum_{i=1}^{n} \lambda_i^k x_{ii}$. As $\lambda_i \neq \lambda_j > 0$,

by $\circledast$ we get the diag. entries of $\gamma\gamma_1\gamma^{-1}$ and $\rho_t(\gamma\gamma_1\gamma^{-1})$

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are the same. So $tr(\gamma\gamma_1\gamma^{-1}) = tr(\rho_t(\gamma\gamma_1\gamma^{-1}))$. Thus $tr(\gamma_1) = tr(\rho_t(\gamma_1))$ for any $\gamma_1 \in \Gamma$. Therefore by Step 1 we are done. ∎