Lecture 6: Reformulation; Stability of $\mathbb{R}$-regularity

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<u>Proposition′</u>. Let C be a compact subset of G, let $\mathcal{O}_G$ be a nbhd of I in $SL_n(\mathbb{R})$, let $\mathcal{O}_A$ be a nbhd of I in A, and $t>1$. Then there is a nbhd U_G of I in $SL_n(\mathbb{R})$ such that for any $k\in C$ and $a\in A_t$ we have

$$kak^{-1}U_G \subseteq k\{gadg^{-1} \mid g\in\mathcal{O}_G, d\in\mathcal{O}_A\}k^{-1}.$$

<u>Proposition′ $\Rightarrow$ Proposition.</u>

Let $\mathcal{O}'_G\subseteq\mathcal{O}_G$ be a nbhd of I such that ① $\mathcal{O}'_G$ is bounded. and ② $\overline{\mathcal{O}'_G}\cdot\mathcal{O}'_G\subseteq\mathcal{O}_G$. Apply <u>Proposition′</u> to $C:=\overline{\mathcal{O}'_G}$, $\mathcal{O}_G:=\mathcal{O}'_G$, $\mathcal{O}_A$, and t, to get a nbhd U'_G of I in $SL_n(\mathbb{R})$ such that $\forall k\in\overline{\mathcal{O}'_G}$, $a\in A_t$,

$$kak^{-1}U'_G\subseteq k\{gadg^{-1}\mid g\in\mathcal{O}'_G, d\in\mathcal{O}_A\}k^{-1}.$$

Let C' be a compact subset of $\mathcal{O}'_G\cap U'_G$ such that $I\in C'^{\circ}$. And

Let $U:=(C')^{\circ}\cap\bigcap_{k\in C'}k^{-1}U'_G$.

<u>Claim</u> U is a nbhd of I in $SL_n(\mathbb{R})$.

<u>Pf of claim.</u> $I\in(C')^{\circ}$ and $\forall k\in C'\subseteq U'_G$, $I\in k^{-1}U'_G$ imply $I\in U$.

If U is NOT open, then $C'\setminus U$ is NOT closed. So $\exists\, x_i\in C'\setminus U$ s.t. $x_i\to x\in U$. Hence, for any i, $\exists\, k_i\in C$ s.t. $k_ix_i\notin U'_G$.

Passing to a subseq. we can assume $k_i\to k\in C$. Since U'_G is open and $k_ix_i\notin U'_G$, $k_ix_i\xrightarrow[i\to\infty]{} kx\notin U'_G$. So $x\notin k^{-1}U'_G$, which contradicts our assumption that $x\in U$. $\square$

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Now, let $U_G := U$. For any $a \in A_t$, $g_1, g_2 \in U_G$,

$$g_1 a g_2 = g_1 a g_1^{-1} g_1 g_2 \in g_1 a g_1^{-1} g_1 U_G \subseteq g_1 a g_1^{-1} U_G'$$

(as $g_1 \in C'$)

$$\subseteq g_1 \{ g a d g^{-1} \mid g \in O_G', d \in O_A \} g_1^{-1}$$

$$= \{ (g_1 g) a d (g_1 g)^{-1} \mid g \in O_G', d \in O_A \}$$

$$\subseteq \{ g a d g^{-1} \mid g \in O_G, d \in O_A \}$$

(as $\overline{O_G'}\, O_G' \subseteq O_G$ and $g_1 \in C' \subseteq O_G'$.)

$$\Rightarrow U_G a U_G \subseteq \{ g a d g^{-1} \mid g \in O_G, d \in O_A \}.$$ ■

<u>Remark</u>. The advantage of <u>Proposition'</u> to <u>Proposition</u> is on the fact that, now we can put "unknown" set on one side and "the rest" on the other side: we have to show

$\exists$ a nbhd U_G of I s.t.

$$\boxed{U_G \subseteq \bigcap_{\substack{k \in C \\ a \in A_t}} k a^{-1} \{ g a d g^{-1} \mid g \in O_G, d \in O_A \} k^{-1}}$$

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<u>Lemma</u>. Let $\mathcal{O}$ be a nbhd of I in G, and let C be a compact a subset of G. Then $\bigcap_{k\in C} k\mathcal{O}k^{-1}$ is a nbhd of I in G.

<u>Pf.</u> If not, $\exists$ $x_i \notin \bigcap_{k\in C} k\mathcal{O}k^{-1}$ s.t. $x_i \longrightarrow x \in \bigcap_{k\in C} k\mathcal{O}k^{-1}$.

So, for any i, $\exists$ $k_i \in C$ s.t. $x_i \notin k_i \mathcal{O} k_i^{-1} \Rightarrow k_i^{-1} x_i k_i \notin \mathcal{O}$

Passing to a subsequence, we can assume $k_i \longrightarrow k \in C$. So

$k_i^{-1} x_i k_i \longrightarrow k^{-1} x k$. Since $\mathcal{O}$ is open and $k_i^{-1} x_i k_i \notin \mathcal{O}$,

we get that $k^{-1} x k \notin \mathcal{O}$. Hence $x \notin k\mathcal{O}k^{-1}$ which contradicts

our assumption that $x \in \bigcap_{k'\in C} k'\mathcal{O}k'^{-1}$. $\square$

Using the above lemma, to prove Proposition', it is enough to show

$$I \in \left(\bigcap_{a\in A_t} a^{-1} \{ g \, \mathrm{ad} \, g^{-1} \mid g \in \mathcal{O}_G, d \in \mathcal{O}_A \} \right)^{\circ}$$

Using the $N^- \times A \times N^+$ decomposition, replacing $\mathcal{O}_G$ by a smaller nbhd of I we can assume $\mathcal{O}_G = \mathcal{O}_{N^-} \mathcal{O}_{N^+} \mathcal{O}'_A$ for some bounded nbhds of I in N^-, N^+, and A. So we need to show:

$$I \in \mathcal{O} := \left(\bigcap_{a\in A_t} a^{-1} \{ n_- n_+ \, \mathrm{ad} \, n_+^{-1} n_-^{-1} \mid n_+ \in \mathcal{O}_{N^-}, n_- \in \mathcal{O}_{N^+}, d \in \mathcal{O}_A \} \right)^{\circ}$$

If it is NOT the case, then $\exists x_i \longrightarrow I$ and $x_i \notin \mathcal{O}$.

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Since $x_i \notin \mathcal{O}$, for any i there is $a_i \in A_+$ s.t.

$$x_i \notin \mathcal{O}(a_i) := a_i^{-1} \{ n_- n_+ a_i d n_+^{-1} n_-^{-1} \mid n_+ \in \mathcal{O}_{N_+}, n_- \in \mathcal{O}_{N_-}, d \in \mathcal{O}_A \}.$$

We can further assume $x_i \in \overline{\mathcal{O}(a_i)}$ as $\mathcal{O}(a_i)$ contains a nbhd of I in $SL_n(\mathbb{R})$. Hence

$$x_i = a_i^{-1} n_-^{(i)} n_+^{(i)} a_i d_i n_+^{(i)-1} n_-^{(i)-1} \text{ for some}$$

$$(n_+^{(i)}, n_-^{(i)}, d_i) \in \overline{\mathcal{O}}_{N_+} \times \overline{\mathcal{O}}_{N_-} \times \overline{\mathcal{O}}_A \setminus \mathcal{O}_{N_+} \times \mathcal{O}_{N_-} \times \mathcal{O}_A$$

Passing to subsequence, we can assume

$$n_+^{(i)} \longrightarrow n_+ \in \overline{\mathcal{O}}_{N_+},\ n_-^{(i)} \longrightarrow n_- \in \overline{\mathcal{O}}_{N_-},\ d_i \longrightarrow d \in \overline{\mathcal{O}}_A$$

As $x_i \longrightarrow I$, we get that $a_i^{-1} n_-^{(i)} n_+^{(i)} a_i \longrightarrow n_- n_+ d$. So

$$a_i^{-1} n_-^{(i)} a_i \cdot a_i^{-1} n_+^{(i)} a_i \longrightarrow n_- n_+ d.$$

Since $n_-^{(i)}$ is bounded and $a_i \in A_+$, $a_i^{-1} n_-^{(i)} a_i$ is bounded.

So passing to a subseq. we can assume $a_i^{-1} n_-^{(i)} a_i \longrightarrow n_-'$.

Hence $\underbrace{a_i^{-1} n_+^{(i)} a_i}_{\in N_+} \longrightarrow n_-'^{-1} n_- n_+ d$.

Therefore by $N_- \times A \times N_+$-decomposition, $d = 1$, $n_-' = n_-$, and $a_i^{-1} n_+^{(i)} a_i \longrightarrow n_+$. Since $a_i \in A_+$, conjugation by a_i

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expands N^+ and contracts N^-. So

$$\left.\begin{array}{l} n_+^{(i)} \longrightarrow n_+ \\ a_i^{-1} n_+^{(i)} a_i \longrightarrow n_+ \end{array}\right\} \Rightarrow n_+ = I \quad \text{and} \quad \left.\begin{array}{l} n_-^{(i)} \longrightarrow n_- \\ a_i^{-1} n_-^{(i)} a_i \longrightarrow n_- \end{array}\right\} \Rightarrow n_- = I.$$

So, for large enough i, $n_+^{(i)} \in \mathcal{O}_{N_+}$, $n_-^{(i)} \in \mathcal{O}_{N_-}$, $d_i \in \mathcal{O}_A$ which contradicts the fact that $(n_+^{(i)}, n_-^{(i)}, d_i) \notin \mathcal{O}_{N_+} \times \mathcal{O}_{N_-} \times \mathcal{O}_A$.