Lecture 3: Chevalley's theorem

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In the proof of Borel's density theorem, we used the following theorem by Chevalley. As I promised, today we will prove this theorem:

<u>Theorem</u> (Chevalley) Let $G \subseteq GL_n(\mathbb{C})$ be a Zariski-closed subgroup and $H \subsetneq G$ be a proper Zariski-closed subgroup. Then $\exists \rho: G \to GL_m(\mathbb{C})$ and $v_0 \in \mathbb{C}^m \setminus \{0\}$ such that

$$H = \{g \in G \mid \rho(g)[v_0] = [v_0]\}$$

where $[v_0] = \mathbb{C} v_0 \in \mathbb{P}(\mathbb{C}^m)$.

We start by defining the ring of regular functions of G.

. Let $\widetilde{I}_G := \{f \in \mathbb{C}[X_{ij}] \mid f(G) = 0\}$. If $f_1, f_2 \in \widetilde{I}_G$, then $\forall f \in \mathbb{C}[X_{ij}]$ and $g \in G$, we have

$$(f_1 + f f_2)(g) = f_1(g) + f(g) f_2(g) = 0$$

$$\Rightarrow f_1 + f f_2 \in \widetilde{I}_G \Rightarrow \widetilde{I}_G \text{ is an ideal of } \mathbb{C}[X_{ij}].$$

<u>Definition</u>. Let $\mathbb{C}[G] := \mathbb{C}[X_{ij}]/\widetilde{I}_G$, and we call it the ring of regular functions of G.

<u>Lemma</u>. $\{f|_G \mid f \in \mathbb{C}[X_{ij}]\} \xrightarrow{\theta} \mathbb{C}[G]$, $f|_G \mapsto f + \widetilde{I}_G$ is a $\mathbb{C}$-algebra isomorphism.

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Proof. Well-defined. $f_1|_G = f_2|_G \Rightarrow (f_1-f_2)|_G = 0 \Rightarrow f_1 - f_2 \in \tilde{I}_G$

$\Rightarrow f_1 + \tilde{I}_G = f_2 + \tilde{I}_G \Rightarrow \Theta(f_1|_G) = \Theta(f_2|_G)$.

Injective. $\Theta(f|_G) = 0 \Rightarrow f + \tilde{I}_G = 0 \Rightarrow f \in \tilde{I}_G$

$\Rightarrow f|_G = 0$.

Surjective. $\forall f \in \mathbb{C}[X_{ij}]$, $f + \tilde{I}_G = \Theta(f|_G)$. ■

Lemma. If $H \subseteq G$, then $\tilde{I}_G \subseteq \tilde{I}_H$.

Proof. $f \in \tilde{I}_G \Rightarrow f|_G = 0 \Rightarrow f|_H = 0 \Rightarrow f \in \tilde{I}_H$. ■

(not needed) [Lemma $f^n \in \tilde{I}_G \Rightarrow f \in \tilde{I}_G$

Pf. $\left(\left. \begin{array}{l} \forall g \in G,\ f(g)^n = 0 \\ \mathbb{C} \text{ has no non-zero nilpotents} \end{array} \right\} \Rightarrow f(g) = 0 \right) \Rightarrow f \in \tilde{I}_G$. ■]

Def.

Let $I_H := \tilde{I}_H / \tilde{I}_G \lhd \mathbb{C}[G]$ and let's call it the defining ideal of H

To prove Chevalley's theorem we construct spaces $(W \subseteq V)$ with the following properties:

(*) ① V is a G-mod.

② $H = \{g \in G \mid gW = W\}$.

Step ① Find (W, V) which satisfies (*) and V is a locally-finite

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G-mod. I.e. $\forall v \in V$, the span of $g.v$ is finite-dimensional.

<u>Step 2</u>. Find (W,V) which satisfies $\circledast$ and $\dim V < \infty$.

<u>Step 3</u>. Find (W,V) which satisfies $\circledast$, $\dim V<\infty$, and $\dim W = 1$.

To get Step 1, we make use of the action of G on $\mathbb{C}[G]$:

$\forall g \in G$, $f \in \mathbb{C}[G]$, let $\lambda_g f : G \to \mathbb{C}$, $(\lambda_g f)(g') = f(g^{-1}g')$.

(Here we have identified $\mathbb{C}[G]$ with $\{f|_G \mid f \in \mathbb{C}[X_{ij}]\}$.

Alternatively we can write

$$\lambda_g(f + \tilde{I}_G) := f(g^{-1}X) + \tilde{I}_G$$

and notice that it is well-defined:

$$f \in \tilde{I}_G \Rightarrow f|_G = 0 \Rightarrow \forall g, g' \in G,\ f(g^{-1}g') = 0$$
$$\Rightarrow \lambda_g(f)|_G = 0 \ .)$$

<u>Lemma</u>. For any $f \in \mathbb{C}[G]$, the G-mod generated by f is finite-dimensional, i.e. the $\mathbb{C}$-span of $\{\lambda_g(f) \mid g \in G\}$ has finite dimension.

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<u>Proof</u>. For any $f \in \mathbb{C}[G]$ and $g \in G$,

$$\lambda_g(f) = f(g^{-1}X) + \tilde{I}_G = f([L_{ij}(\underline{X})]) + \tilde{I}_G$$

$$\underbrace{\in \{p \in \mathbb{C}[X_{ij}] \mid \deg_{x_{rs}} p \leq \deg_{x_{rs}} f\} + \tilde{I}_G}_{\text{has finite dimension}}$$

where L_{ij}'s are linear functions

has finite dimension. ∎

<u>Lemma</u>. $H = \{g \in G \mid \lambda_g(I_H) = I_H\}$.

<u>Proof</u>. • $h \in H$, $f \in I_H \Rightarrow \forall h' \in H$, $(\lambda_h f)(h') = f(h^{-1}h') = 0$

$\left(\text{as } f|_H = 0 \text{ and } h^{-1}h' \in H\right)$

$\Rightarrow \lambda_h f|_H = 0 \Rightarrow \lambda_h f \in I_H$.

This implies $\lambda_h(I_H) \subseteq I_H$ for any $h \in H$. So $\lambda_{h^{-1}}(I_H) \subseteq I_H$.

Therefore $\lambda_h(I_H) = I_H$ for any $h \in H$. Hence LHS $\subseteq$ RHS.

• Suppose $\lambda_g(I_H) = I_H$ and $f \in I_H$. Then

$\lambda_{g^{-1}}(f) \in I_H \Rightarrow \lambda_{g^{-1}} f|_H = 0 \Rightarrow \lambda_{g^{-1}} f\ (I) = 0 \Rightarrow f(g) = 0$.

$\Rightarrow g \in H \rightarrow$ RHS $\subseteq$ LHS. ∎

(This gives us the first step.)

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<u>Lemma</u>. There is a finite-dimensional G-mod V and a subspace W of V such that $\{g\in G \mid gW=W\}=H$.

Proof. Since $\mathbb{C}[G]$ is a finitely generated $\mathbb{C}$-algebra, it is Noetherian. Hence for some $f_1,\ldots,f_\ell\in\mathbb{C}[G]$ we have $I_H=\langle f_1,\ldots,f_\ell\rangle$ (as an ideal).

Let V be the G-submodule of $\mathbb{C}[G]$ generated by $\overline{x_{ij}}:=x_{ij}+I_H$ and f_i's. And let W be the H-module generated by f_i's.

- Since $\mathbb{C}[G]$ is a locally finite G-mod, $\dim_{\mathbb{C}} V<\infty$.
- Since W is an H-mod., $H\subseteq\{g\in G\mid gW=W\}$.
- Since I_H is H-invariant, $W\subseteq I_H$. Hence the ideal generated by W is I_H.
- Notice that $G\hookrightarrow \mathrm{Aut}_{\mathbb{C}\text{-alg.}}(\mathbb{C}[G])$. So, for any $g\in G$, the ideal generated by $\lambda_g(W)=\lambda_g$(the ideal generated by W). Hence, if $\lambda_g(W)=W$, then $I_H=\lambda_g(I_H)$. Therefore $g\in H$. ∎

Lecture 3: Wedge powers of a vector space

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<u>Definition</u>. Let V be an m-dimensional vector space over $\mathbb{C}$.

The d^{th} wedge power of V is denoted by $\Lambda^n V$, and it is defined as

$$V \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} V \Big/ \underbrace{\mathbb{C}\text{-span of } \{v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \mid v_i \in V\}}_{S(V)}.$$

For v_i's in V, $v_1 \otimes \cdots \otimes v_n + S(V)$ is denoted by $v_1 \wedge \cdots \wedge v_n$.

<u>Basic properties of $\Lambda^n V$.</u>

① For any permutation $\sigma \in S_n$,

$$v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)} = \operatorname{Sgn}(\sigma)\, v_1 \wedge \cdots \wedge v_n.$$

② $v_1 \wedge v_2 \wedge \cdots \wedge v_n = 0$ if $v_i = v_j$ for some $i \neq j$.

③ For any $c, d \in \mathbb{C}$,

$$v_1 \wedge \cdots \wedge (c v_i + d v_i') \wedge \cdots \wedge v_n = c\, v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_n + d\, v_1 \wedge \cdots \wedge v_i' \wedge \cdots \wedge v_n.$$

④ If $\{e_1, \ldots, e_m\}$ is a $\mathbb{C}$-basis of V, then

$$\{e_{i_1} \wedge \cdots \wedge e_{i_n} \mid i_1 < i_2 < \cdots < i_n\} \text{ is a } \mathbb{C}\text{-basis of } \Lambda^n V.$$

So $\dim_{\mathbb{C}} \Lambda^n V = \binom{m}{n}$; in particular, $\dim_{\mathbb{C}} \Lambda^n V = 0$ if $n > m$, and $\dim_{\mathbb{C}} \Lambda^m V = 1$.

Lecture 3: Wedge powers of a vector space

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__Lemma.__ Let $C = \begin{bmatrix} c_{11} & \cdots & c_{1d} \\ \vdots & \vdots & \vdots \\ c_{n1} & \cdots & c_{nd} \end{bmatrix} \in M_{n\times d}(\mathbb{C})$. Then we have

$$\left(\sum_{i=1}^{n} c_{i1} v_i\right) \wedge \left(\sum_{i=1}^{n} c_{i2} v_i\right) \wedge \cdots \wedge \left(\sum_{i=1}^{n} c_{id} v_i\right) =$$

$$\sum_{\substack{I=\{i_1,\dots,i_d\} \\ 1 \le i_1 < i_2 < \cdots < i_d \le n}} \det(C_I)\, v_{i_1} \wedge \cdots \wedge v_{i_d},$$

where $C_I = \begin{bmatrix} c_{i_1 1} & \cdots & c_{i_1 d} \\ \vdots & \ddots & \vdots \\ c_{i_d 1} & \cdots & c_{i_d d} \end{bmatrix}$.

__Proof.__ $$\left(\sum_{i=1}^{n} c_{i1} v_i\right) \wedge \cdots \wedge \left(\sum_{i=1}^{n} c_{id} v_i\right) \overset{\text{multi-linear}}{=} \sum_{1 \le i_1,\dots,i_d \le n} \left(\prod_{j=1}^{d} c_{i_j j}\right) v_{i_1} \wedge \cdots \wedge v_{i_d}$$

$$= \sum_{\substack{1 \le i_1,\dots,i_d \le n \\ (i_l \ne i_s \text{ if } l \ne s)}} \left(\prod_{j=1}^{d} c_{i_j j}\right) v_{i_1} \wedge \cdots \wedge v_{i_d}$$

$$= \sum_{\substack{I \subseteq \{1,\dots,n\} \\ |I|=d}} \left(\sum_{\sigma \in S_d} \left(\prod_{j=1}^{d} c_{i_{\sigma(j)} j}\right) v_{i_{\sigma(1)}} \wedge \cdots \wedge v_{i_{\sigma(d)}}\right)$$

$$= \sum_{\substack{I \subseteq \{1,\dots,n\} \\ |I|=d}} \underbrace{\left(\sum_{\sigma \in S_d} \operatorname{sgn}(\sigma) \prod_{j=1}^{d} c_{i_{\sigma(j)} j}\right)}_{\det C_I} v_{i_1} \wedge \cdots \wedge v_{i_d}.$$

∎

__Def./lem.__ For $x \in M_n(\mathbb{C})$ and $1 \le d \le n$, x (naturally) acts on $\Lambda^d \mathbb{C}^n$: for any $v_1,\dots,v_d \in \mathbb{C}^n$,

$$x \cdot (v_1 \wedge \cdots \wedge v_d) := (x v_1) \wedge \cdots \wedge (x v_d).$$

Lecture 3: Action on the wedge space and the final step

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To show that this is a well-defined $\mathbb{C}$-linear map, first one can use the universal property of tensor product to show

$$x\cdot(v_1\otimes\cdots\otimes v_d):=(xv_1)\otimes\cdots\otimes(xv_d)$$

is well-defined. Then one can easily check that $x\cdot S(V)\subseteq S(V)$. Hence the above map is well-defined (and linear). $\square$

<u>Lemma</u>. Suppose W is a proper non-trivial subspace of a finite-dimensional space V. Then

$$\{g\in GL(V)\mid gW=W\}=\{g\in GL(V)\mid g\cdot \textstyle\bigwedge^d W=\bigwedge^d W\}$$

where $d=\dim_{\mathbb{C}} W$.

<u>Proof</u>. Let $\{e_1,\ldots,e_d\}$ be a $\mathbb{C}$-basis of W, and $\{e_1,\ldots,e_d,\ldots,e_n\}$ be a $\mathbb{C}$-basis of V. Then $\{e_{i_1}\wedge\cdots\wedge e_{i_d}\mid i_1<\cdots<i_d\}$ is a $\mathbb{C}$-basis of V and $W=\mathbb{C}(e_1\wedge\cdots\wedge e_d)$.

- $$\left.\begin{array}{l} gW=W \Rightarrow g\cdot\bigwedge^d W\subseteq \bigwedge^d W \\ \Downarrow \\ g^{-1}W=W\Rightarrow g^{-1}\cdot\bigwedge^d W\subseteq\bigwedge^d W\end{array}\right\}\Rightarrow g\cdot\textstyle\bigwedge^d W=\bigwedge^d W.$$
- Suppose $gW\neq W$ and $ge_j=\sum_{i=1}^n g_{ij}e_i$.

Lecture 3: The final step

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Since $\begin{bmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{bmatrix}$ is invertible, rank $\begin{bmatrix} g_{11} & \cdots & g_{1d} \\ \vdots & \vdots & \vdots \\ g_{n1} & \cdots & g_{nd} \end{bmatrix} = d$. So there are

$I \subseteq \{1, \dots, n\}$ and $|I| = d$ such that $\det g_I \neq 0$. If $\det g_{I_0} \neq 0$ for some $I_0 \neq \{1, \dots, d\}$, then

$$g \cdot (e_1 \wedge \cdots \wedge e_d) = \sum_{\substack{I \subseteq \{1, \dots, n\} \\ I = \{i_1 < \cdots < i_d\}}} \det(g_I)\, e_{i_1} \wedge \cdots \wedge e_{i_d}$$

$$= \cdots + \det(g_{I_0})\, e_{i_1^{(0)}} \wedge \cdots \wedge e_{i_d^{(0)}} + \cdots \notin \mathbb{C}\, e_1 \wedge \cdots \wedge e_d = \bigwedge^d W.$$

Hence, if $gW \neq W$ and $g \cdot \bigwedge^d W = \bigwedge^d W$, then

① $\det g_{\{1, \dots, d\}} \neq 0$

② $\det g_I = 0$ if $I \subseteq \{1, \dots, n\}$, $|I| = d$, and $I \neq \{1, \dots, d\}$.

③ $\exists\, i_0$, $d < i_0 \leq n$ and $[g_{i_0 1} \cdots g_{i_0 d}] \neq \vec{0}$.

Since $\det g_{\{1, \dots, d\}} \neq 0$, $\exists\, c_1, \dots, c_d \in \mathbb{C}$ s.t.

$$[g_{i_0 1} \cdots g_{i_0 d}] = c_1 [g_{11} \cdots g_{1d}] + \cdots + c_d [g_{d1} \cdots g_{dd}].$$

So for some j_0, $c_{j_0} \neq 0$. Therefore

$$[g_{j_0 1} \cdots g_{j_0 d}] \in \mathrm{Span}\big([g_{11} \cdots g_{1d}], \dots, \underbrace{\widehat{[g_{j_0 1} \cdots g_{j_0 d}]}}_{}, \dots, [g_{d1} \cdots g_{dd}],$$

$$[g_{i_0 1} \cdots g_{i_0 d}]\big).$$

(remove)

Hence $\det g_I \neq 0$ where $I = (\{1, \dots, d\} \setminus \{j_0\}) \cup \{i_0\}$, which contradicts ②. ∎

Lecture 3: The final step

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We have already found (W, V) such that

① V is a finite-dimensional G-mod.

② $\{g \in G \mid gW = W\} = H$.

Now consider $(\overset{d}{\wedge} W, \overset{d}{\wedge} V)$, where $d = \dim_{\mathbb{C}} W$:

① $\overset{d}{\wedge} V$ is a finite-dimensional G-mod.

② $\dim_{\mathbb{C}} \overset{d}{\wedge} W = 1$ (so it is a line).

③ $\{g \in G \mid g \cdot \overset{d}{\wedge} W = \overset{d}{\wedge} W\} = \{g \in G \mid gW = W\} = H$. ∎