Lecture 1: Number theory; Irrational rotations.

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11:02 AM

I. Counting integral or rational points

$$X := \{\vec{x} \mid f_1(\vec{x}) = f_2(\vec{x}) = \cdots = f_m(\vec{x}) = 0\}$$

Affine Variety.

$$N_T(X) := |\{\vec{x} \in X(\mathbb{Z}) \mid \|\vec{x}\| \leq T\}|$$

Examples: $X = SL_n$.

X : Homogeneous Variety i.e. G/H.
we have a good chance.

As you well know in general $X(\mathbb{Z})$ might be very small compared to $X(\mathbb{C})$. Having a rich group action can help us to get lots of solutions from a "single" solution.

II. Oppenheim Conj. (1929)

$Q(\vec{x})$ a quadratic form, not porpotionate to a rational form, not positive-definite (or neg. definite) dim ≥ 3.

$$\Rightarrow \overline{Q(\mathbb{Z}^3)} = \mathbb{R}.$$

III. Littlewood Conj. (1930)

$$\alpha, \beta \in \mathbb{R} \Rightarrow \liminf_{n\to\infty} n\{n\alpha\}\{n\beta\} = 0.$$

Dynamical Side:

X : either a topological space or a measure space.

$T: X \to X$ a "nice" transformation.

(a semigroup action $\mathbb{Z}^{\geq 0}$ or in general a group action $H \curvearrowright X$.)

① What can we say about the orbit closure of any point? How about a "typical point"?

② Can we understand how an orbit is distributed?

For a "nice" function f, does

$$\frac{1}{N} \sum_{i=0}^{N-1} f(T^i x)$$

converges? How does the limit depend on x?

Exp. $T_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, $T_\alpha(x+\mathbb{Z}) = \alpha + x + \mathbb{Z}$.

$\alpha \notin \mathbb{Q} \implies$ any orbit is equidistributed:

$$\forall f \in C(\mathbb{R}/\mathbb{Z}), \qquad \frac{1}{N}\sum_{i=0}^{N-1} f(T_\alpha^i x_0) \xrightarrow[N\to\infty]{} \int_0^1 f(t)\,dt$$

Proof.

Let $F_N(x) = \frac{1}{N}\sum_{i=0}^{N-1} f(i\alpha + x)$. Instead of pointwise convergence, we start with weak convergence, i.e.

$$\forall g \in L^2(\mathbb{R}/\mathbb{Z}), \quad \langle F_N, g\rangle \xrightarrow[N\to\infty]{} \langle \mathbb{1}, f\rangle \langle \mathbb{1}, g\rangle \qquad (*)$$

We know that $\{e^{2\pi i k t}\}_{k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R}/\mathbb{Z})$. So it is enough to show $(*)$ for these "test" functions.

$$\langle F_N, \chi_k\rangle = \frac{1}{N}\sum_{j=0}^{N-1}\int_0^1 f(j\alpha+t)\, e^{2\pi i k t}\,dt$$

$$= \frac{1}{N}\sum_{j=0}^{N-1}\int_0^1 f(t')\, e^{2\pi i k(t'-j\alpha)}\,dt$$

$$= \frac{\langle f, \chi_k\rangle}{N}\sum_{j=0}^{N-1}\left(e^{-2\pi i k\alpha}\right)^j$$

$$= \frac{\quad}{N} \sum_{j=0} (e \quad)$$

$k \neq 0$

Since $\alpha \notin \mathbb{Q}$

$$= \langle f, \chi_k \rangle \frac{\left(e^{-2\pi i k\alpha}\right)^N - 1}{N\left(e^{-2\pi i k\alpha} - 1\right)}$$

$$\xrightarrow[N \to \infty]{} 0$$

$\chi_0 = \mathbb{1}$ and so $\circledast$ is proved.

<u>Weak Convergence To Pointwise Convergence</u>

Let $\{\psi_\varepsilon\}$ be a family of "test" functions converging δ_{x_0}

i.e., ψ_ε are C^∞ ; $\psi_\varepsilon \geq 0$; $\langle \psi_\varepsilon, \mathbb{1} \rangle = 1$.

- $\operatorname{supp} \psi_\varepsilon \subseteq (-\varepsilon + x_0, \varepsilon + x_0)$ (w.l.o.g. suppose $0 < x_0 < 1$.)
- $\langle \psi_\varepsilon, f \rangle \xrightarrow[\varepsilon \to 0]{} f(x_0)$ for any continuous function f.

(in the circle)

For any $\varepsilon > 0$, $\exists\, \delta > 0$, s.t. if $|x - y| < \delta$, then

$$|f(x) - f(y)| < \varepsilon.$$

(unif. cont.)

$$\Rightarrow \quad |F_N(x) - F_N(y)| \leq \frac{1}{N} \sum_{j=0}^{N-1} |f(j\alpha + x) - f(j\alpha + y)|$$

$$\leq \varepsilon$$

if $|x-y| \le \delta$.

$$\Rightarrow \quad F_N(x_0) - \varepsilon \le \langle F_N, \psi_{\delta'} \rangle \le F_N(x_0) + \varepsilon$$

for any $\delta' \ll_\varepsilon 1$.

On the other hand, $\langle F_N, \psi_{\delta'} \rangle \xrightarrow[N\to\infty]{} \langle f, \mathbb{1} \rangle$.

So, if $N \gg_{\delta'} 1$, we have

$$F_N(x_0) - 2\varepsilon \le \langle f, \mathbb{1} \rangle \le F_N(x_0) + 2\varepsilon$$

$$\Rightarrow \quad F_N(x_0) \xrightarrow[N\to\infty]{} \langle f, \mathbb{1} \rangle. \qquad \blacksquare$$

<u>Corollary</u>. $\frac{1}{N} \left| \{ n \in [0, N-1] \mid \{n\alpha\} \in (a,b) \} \right| \xrightarrow[N\to\infty]{} b-a$ if $\alpha \notin \mathbb{Q}$.

<u>Pf.</u> Let f_ε^+ be a continuous approximation of (a,b), i.e.

$$f_\varepsilon^+\big|_{[a,b]} = 1$$

$$f_\varepsilon^+\big|_{[0,1]\setminus(a-\varepsilon, b+\varepsilon)} = 0$$

$$0 \le f(t) \le 1.$$

$$\Rightarrow \frac{\left| \{ n \in [0, N-1] \mid \{n\alpha\} \in (a,b) \} \right|}{N} = \frac{1}{N} \sum_{i=0}^{N-1} \mathbb{I}_{(a,b)}(T_\alpha^i(0))$$

and

$$\frac{1}{N} \sum^{N-1} f_\varepsilon^-(T_\alpha^i(0)) \le \frac{1}{N} \sum^{N-1} \mathbb{I}_{(a,b)}(T_\alpha^i(0)) \le \frac{1}{N} \sum^{N-1} f_\varepsilon^+(T_\alpha^i(0))$$

$$\frac{1}{N}\sum_{i=0}^{N-1} f_\varepsilon^-(T_\alpha^i(0)) \le \frac{1}{N}\sum_{i=0}^{N-1} I_{(a,b)}(T_\alpha^i(0)) \le \frac{1}{N}\sum_{i=0}^{N-1} f_\varepsilon^+(T_\alpha^i(0))$$

$$\Rightarrow \quad \langle f_\varepsilon^-, \mathbb{1}\rangle \le \underline{\lim}\, a_N \le \overline{\lim}\, a_N \le \langle f_\varepsilon^+, \mathbb{1}\rangle$$

Let $\varepsilon \to 0$; $\langle f_\varepsilon^\pm, \mathbb{1}\rangle \longrightarrow b-a$. So we are done. $\blacksquare$