Unit Tangent bundle of $\mathbb{H}^2$.

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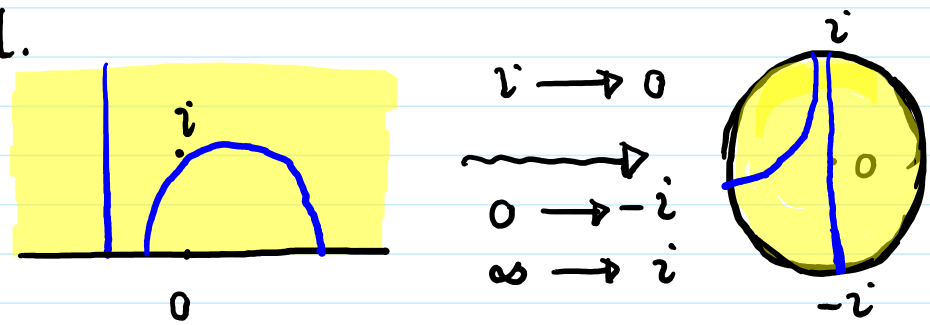
We know that $PSL_2(\mathbb{R}) \simeq \mathrm{Isom}(\mathbb{H}^2)^0$. If we use the upper-half plane model of $\mathbb{H}^2$, then $PSL_2(\mathbb{R})$ acts via Möbius transformations. Sometimes it is useful to conj. this action by a conformal map and use the Poincaré disc model.



In both of these models geodesics are (parts of) either circles or lines that are orthogonal to the boundary.
And the action of $\mathrm{Isom}(\mathbb{H}^2)$ extends uniquely to the boundary.

$\alpha(t) := \begin{bmatrix} e^{t/2} & \\ & e^{-t/2} \end{bmatrix} = \exp\left(t \begin{bmatrix} 1/2 & \\ & -1/2 \end{bmatrix}\right)$ acting on i

(and the origin in the Poincaré disc model) gives us a length parametrization of the geodesic going from 0 to ∞ (resp., $-i$ to i) where the initial point is i (resp., 0).

- If an orientation preserving isometry fixes a geodesic pointwise, then it is identity.

- $\forall z \in \mathbb{H}^2, \xi \in \partial \mathbb{H}^2$, $\exists !$ geodesic which passes through z and has ξ as one of its end points.
- $\forall z_1, z_2 \in \mathbb{H}^2, \xi_1, \xi_2 \in \partial \mathbb{H}^2$, $\exists$ an orientation preserving isometry g s.t.

$$g(z_1) = z_2 \quad \text{and} \quad g(\xi_1) = \xi_2$$

and so it sends the geodesic connecting z_1 to ξ_1 to the geodesic connecting z_2 to ξ_2. Hence by the above remarks it is unique.

Summary. $PSL_2(\mathbb{R}) \xrightarrow{\Theta} \mathbb{H}^2 \times \partial \mathbb{H}^2$

$$g \longmapsto (g(z_0), g(\xi_0))$$

, where $(z_0, \xi_0) \in \mathbb{H}^2 \times \partial \mathbb{H}^2$ is a fixed point, is a bijection.

Clearly Θ is continuous. And it is straight-forward to see that it is a homeomorphism.

We can identify $\mathbb{H}^2 \times \partial \mathbb{H}^2$ with the unit tangent bundle $T^1(\mathbb{H}^2)$ of $\mathbb{H}^2$. And Θ is a $PSL_2(\mathbb{R})$-equivariant map, i.e.

$$\forall g_1, g_2 \in PSL_2(\mathbb{R}), \qquad g_1 \cdot \Theta(g_2) = \Theta(g_1 g_2).$$

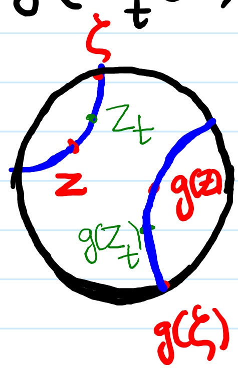
Def. The geodesic flow $a_t : T^1(\mathbb{H}^2) \to T^1(\mathbb{H}^2)$ is defined as follows:

$$a_t(z,\xi) = (z_t, \xi)$$

where z_t is on the geodesic connecting z to ξ, $d(z,z_t) = |t|$, and z_t is between z and ξ if $t > 0$ (z is between z_t and ξ if $t < 0$).

- $\forall g \in \mathrm{Isom}(\mathbb{H}^2)$, $(z,\xi) \in \mathbb{H}^2 \times \partial\mathbb{H}^2$ we have

$$g(a_t(z,\xi)) = (g(z_t), g(\xi))$$



$$a_t(g(z), g(\xi)) = (g(z_t), g(\xi))$$

So $g \circ a_t = a_t \circ g$.

$$\Rightarrow \quad a_t(\theta(g)) = a_t(g(z_0), g(\xi_0))$$
$$= g \cdot a_t(z_0, \xi_0)$$
$$= g\,\alpha(t)\,(z_0, \xi_0)$$
$$= \theta(g\,\alpha(t)).$$

So under the above identification of $PSL_2(\mathbb{R})$ and $T^1(\mathbb{H}^2)$ the geodesic flow is just multiplication by the diag. matrix $\alpha(t) = \exp\left(t \begin{bmatrix} 1/2 & \\ & -1/2 \end{bmatrix}\right)$.

<u>The horocycle flow</u> $u_t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ acts from right on

$PSL_2(\mathbb{R})$. This is called the horocycle flow. Let's see how we can visualize it under the above identification of $PSL_2(\mathbb{R})$ with $T^1(\mathbb{H}^2)$:

$$\mathcal{U}_t(g(z_0), g(\zeta_0)) := \Theta(g\,u_t) = (g u_t(z_0), g u_t(\zeta_0))$$

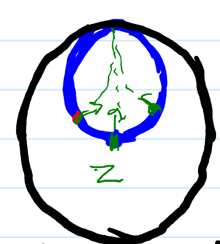
Let's us the upper-half plane model to understand $u_t(z_0)$ and $u_t(\zeta_0)$: in this model $z_0 = i$ and $\zeta_0 = \infty \Rightarrow$

$$u_t(i) = i + t \quad \text{and} \quad u_t(\infty) = \infty.$$

So going back to the Poincaré model the orbit of $(0, i)$ is

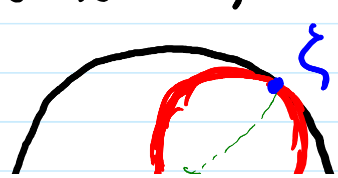
$$\{(z, i) \mid z \text{ is on the circle which passes through } \underline{0}$$
$$\text{and is tangent to the boundary at } i\}.$$

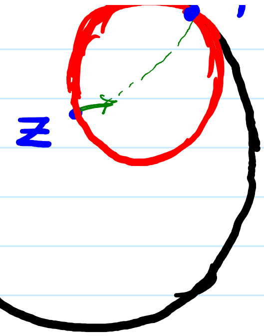
Hence the orbit of the horocycle flow at $(z, \zeta) = (g(z_0), g(\zeta_0))$



is $\{g(u_t(z_0), \zeta_0)\}_{t \in \mathbb{R}} = \{(g(z'), \zeta) \mid z' \text{ is on the above circle}\}$

Since g is an isometry, g of the above circle is a circle which passes through $g(z_0) = z$ and is tangent to the boundary at $g(\zeta_0) = \zeta$.





• Notice that $\mathbb{H}^2$ can be identified with $SL_2(\mathbb{R})/_{SO_2(\mathbb{R})}$.
Now, if $\Gamma \subseteq SL_2(\mathbb{R})$ is a discrete group, then

$_\Gamma\backslash^{\mathbb{H}^2}$ is a hyperbolic orbifold and $T^1(_\Gamma\backslash^{\mathbb{H}^2})$ can

be identified with $_\Gamma\backslash^{SL_2(\mathbb{R})}$ and the geodesic flow

on $T^1(_\Gamma\backslash^{\mathbb{H}^2})$ is the same as $x \longmapsto x\,\alpha(t)$ for any $x \in {_\Gamma\backslash^{SL_2(\mathbb{R})}}$ (Everything can be uniquely extended to the covering space.) And similarly we can define the horocycle flow on $T^1(_\Gamma\backslash^{\mathbb{H}^2})$.

• In the identification, $\mathrm{Isom}(\mathbb{H}^2)^\circ$ acts on $T^1(\mathbb{H}^2)$ from left and the geodesic flow got identified with the action of diag. matrices A from right on $G = PSL_2(\mathbb{R})$. That is another way of seeing that the geodesic flow commutes with isometries:

$$G \curvearrowright G \overset{\text{from right}}{\curvearrowleft} A.$$

from left