Haar measure on quotient spaces.

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10:00 AM

Thm (Haar, von Neumann, Weil)

σ-compact.
G: second countable, Hausdorff, locally compact

$\Rightarrow$ $\exists$ a left-invariant regular measure λ_G on G.

And it is unique up to a constant.

The same is true for right-invariant: ρ_G.

Def. A measure μ is called a regular Borel measure if

① it is defined on the σ-algebra generated by open sets, and

② $\forall\, C \subseteq G$ compact, $\mu(C) < \infty$.

③ $\mu(A) = \sup\{\mu(C) \mid \underset{\text{compact}}{C \subseteq A}\}$ (inner regular)

④ $\mu(A) = \inf\{\mu(O) \mid A \subseteq O \text{ open}\}$ (outer regular)

$\boxed{Q}$ Is there a relation between λ_G and ρ_G?

$\boxed{A1}$ Let $g \in G$. Let μ be the following Borel measure:

$$\mu(A) := \lambda_G(Ag).$$

It is clear that μ is also a left Haar measure.

So $\exists\, \Delta_G(g) \in \mathbb{R}^+$ s.t. $\lambda_G(Ag) = \Delta_G(g)\,\lambda_G(A)$.

for any Borel set A.

It is easy to see $\Delta_G : G \to \mathbb{R}^+$ is a continuous group homomorphism. (Continu. is a consequence of regularity)

Cor. If $\mathrm{Hom}_c(G, \mathbb{R}^+) = \{1\}$, then $\lambda_G = \rho_G$.

Cor. $G/\overline{[G,G]}$ compact $\Longrightarrow \lambda_G = \rho_G$.

(G: semisimple Lie gps; Compact groups.)

Def. A group is called unimodular if $\lambda_G = \rho_G$.

$\boxed{A2}$ Let $\mu : C_c(G) \to \mathbb{C}$,

$$\mu(f) := \int_G f(g)\, \Delta_G(g)^{-1}\, d\lambda_G(g).$$

$\forall g' \in G$, $(\mu . g')(f) = \mu(r_{g'}(f))$,

where $r_{g'}(f)(x) := f(xg')$.

$\Longrightarrow (\mu . g')(f) = \int f(g g')\, \Delta(g)^{-1}\, d\lambda(g)$

$$\Longrightarrow (\mu.g')(f) = \int_G f(g g') \Delta_G(g)^{-1} d\lambda_G(g)$$

$$= \int_G f(g'') \Delta_G(g'' g'^{-1})^{-1} d\lambda_G(g'' g'^{-1})$$

$$= \int_G f(g'') \Delta_G(g'')^{-1} \Delta_G(g') \quad d(\lambda_G \cdot g'^{-1})(g'')$$

$$= \int_G f(g'') \Delta_G(g'')^{-1} \Delta_G(g') \Delta_G(g'^{-1}) d\lambda_G(g'')$$

$$= \mu(f).$$

The Riesz Representation Theorem $\quad X$: σ-compact, (as above)

ℓ : positive linear functional functional on $C_c(X)$

$$\Downarrow$$

$\exists!$ regular measure μ s.t.

$$\ell(f) = \int_X f \, d\mu.$$

Moreover $\quad \underbrace{\mu(U)}_{\text{open}} = \sup\{\ell(f) \mid f \in C_c(X),\ 0 \le f \le 1,\ \operatorname{supp}(f) \subseteq U\}$

and $\quad \underbrace{\mu(C)}_{\text{compact.}} = \inf\{\ell(f) \mid f \in C_c(X),\ f \ge \mathbb{1}_C\}$

So $\quad \Delta(g)^{-1} d\lambda_G(g) = d\rho_G(g).$

In general it is NOT easy to write an explicit Haar measure for computational purposes.

Theorem. Let G be a C^1-group. Suppose G can be identified by an open subset of $\mathbb{R}^n$ (by C^1-homeo.). For any $g \in G$, let $l_g : G \to G$ $l_g(g') = g \cdot g'$. Then
$$\mu(f) = \int_G f(x_1, \dots, x_n) \frac{dx_1\, dx_2 \cdots dx_n}{|\det (d l_x)_e|}$$
defines a left Haar measure, where $e \in G$ is the iden.

Remark.
$$\begin{array}{ccc} G & \xrightarrow{l_g} & G \\ \text{open} \ \cap & & \cap \ \text{open} \\ \mathbb{R}^n & & \mathbb{R}^n \end{array}$$
is a C^1-map

$\Rightarrow$ For any $\vec{y} \in G$,
$$(d\, l_g)_{\vec{y}} : \mathbb{R}^n \to \mathbb{R}^n \quad \text{is a linear map.}$$

Pf. Regularity is clear. So it is enough to check left invarience.

$$(g_0 \cdot \mu)(f) = \mu(\text{ left shift of } f \text{ by } g_0^{-1})$$

$$= \int_G f(\ell_{g_0^{-1}}(x)) \frac{dx}{|\det (d\ell_x)_e|}$$

$$= \int_G f(x') \frac{d(\ell_{g_0}(x'))}{|\det (d\,\ell_{g_0 x'})_e|}$$

$\boxed{x = g_0 x'}$

$$= \int_G f(x') \frac{|\det (d\ell_{g_0})_{x'}| \; dx'}{|\det (d\ell_{g_0})_{x'}| \, |\det (d\ell_{x'})_e|}$$

$\boxed{\text{chain rule Jacobian}}$

$$= \int_G f(x') \frac{dx'}{|\det (d\ell_{x'})_e|}.$$

$$= \mu(f). \qquad \blacksquare$$

<u>Cor.</u> $A = \mathbb{R}^+ \times \cdots \times \mathbb{R}^+ \Rightarrow da = \frac{dx_1}{x_1} \cdot \frac{dx_2}{x_2} \cdots \frac{dx_n}{x_n}.$

<u>Cor.</u> $N = \left\{ \begin{bmatrix} 1 & & n_{ij} \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \middle| \; n_{ij} \in \mathbb{R} \right\}$

$$\Rightarrow dn = dx_{12}\, dx_{13} \cdots dx_{n-1,n}.$$

<u>Pf</u>. It is enough to show $\left| \det (d\ell_n)_I \right| = 1$

for any $n \in N$.

$$(\ell_n(x))_{ij} = (nx)_{ij} = \sum_{t=1}^{m} n_{it}\, x_{tj}$$

$$= x_{ij} + \sum^{m} n_{it}\, x_{tj}.$$

$$= x_{ij} + \sum_{t=i+1}^{m} n_{it}\, x_{tj}.$$

$\Rightarrow \ell_n$ is a linear map and in the following

basis

$$x_{12}, x_{13}, \cdots, x_{1m}, x_{23}, x_{24}, \cdots, x_{2m}, \cdots, x_{m-1,m}$$

$\left[\, (i,j) \precsim (i',j') \text{ if } i<i' \text{ \underline{or} } (i=i' \text{ and } j \leq j). \,\right]$

ℓ_n is lower triangular with 1's on the diagonal. ∎

$$\boxed{\underline{\text{Cor.}}\quad G = GL_n(\mathbb{R}) \Rightarrow dg = \frac{dx_{11}\, dx_{12} \cdots dx_{nn}}{|\det x|^n}.}$$

__Pf__. It is enough to prove

$$|\det d\ell_g| = |\det g|^n.$$

$$(\ell_g(x))_{ij} = \sum_{t=1}^{n} g_{it}\, x_{tj} \quad \text{is a linear map}$$

__basis__ $\xrightarrow{2}$

$$\begin{array}{cccccccccc}
x_{11} & x_{21} & \cdots & x_{n1} & x_{12} & \cdots & x_{n2} & \cdots & x_{1n} \cdots x_{nn} \\
\hline
g_{11} & g_{21} & \cdots & g_{n1} & & & & \\
g_{12} & g_{22} & \cdots & g_{n2} & & & & \\
\vdots & \vdots & & \vdots & & & & \\
g_{1n} & g_{2n} & \cdots & g_{nn} & & & & \\
 & & & & g_{11} & \cdots & g_{n1} & \\
 & & & & \vdots & \ddots & \vdots & \\
 & & & & g_{1n} & \cdots & g_{nn} &
\end{array}$$

$$
\begin{array}{c|c|c|c}
 & g_{1n} \cdots g_{nn} & & \\
\hline
 & & & \\
\hline
 & & & \begin{matrix} g_{11} & \cdots & g_{n1} \\ \vdots & \ddots & \vdots \\ g_{1n} & \cdots & g_{nn} \end{matrix}
\end{array}
$$

$$= \begin{bmatrix} g^t & & \\ & \ddots & \\ & & g^t \end{bmatrix} \Longrightarrow \text{its det} = \det(g)^n. \qquad \blacksquare$$

<u>Remark.</u> A similar argument shows $\dfrac{dx}{|\det(dr_x)|_e}$ is a right Haar measure if $G \hookrightarrow \mathbb{R}^n$ open C^1-embedding.

$$\Big[\ \ (\mu . g)(f) = \mu(r_g(f)) = \int_G f(xg)\, \frac{dx}{|\det(dr_x)_e|}$$

$$\boxed{x' = xg} \quad = \int_G f(x')\, \frac{d(r_{g^{-1}}(x'))}{|\det(dr_{xg^{-1}})_e|}$$

$$\boxed{\begin{array}{l} r_{xg^{-1}} = r_{g^{-1}} \circ r_{x'} \\ \text{Chain Rule} \\ \text{Jacobian} \end{array}} \quad = \int_G f(x')\, \frac{|\det(d\, r_{g^{-1}})_{x'}|\ \ dx'}{|\det(d\, r_{g^{-1}})_{x'}|\, |\det(dr_{x'})_e|}$$

$$= \int_G f(x')\, \frac{dx'}{|\det(dr_{x'})_e|}$$

$$= \int_G f(x') \frac{dx'}{|\det(dr_{x'})_e|}$$

$$= \mu(f).$$]

Now it is easy to see that A, N and $GL_n(\mathbb{R})$ (in the previous corollaries) are unimodular.

Lemma. Let G and H be two unimodular groups. Suppose $\theta: G \to \mathrm{Aut}(H)$ is a group homomorphism and the induced (left) action is continuous, i.e.

$$G \times H \to H, \quad (g,h) \mapsto \theta(g)(h)$$

is continuous. So $G \ltimes H$ is a topological group. Then

- $dg\, dh$ is a left Haar measure on $G \ltimes H$.
- $J_\theta(g)^{-1} dg\, dh$, where $(\theta(g))^*(\lambda_H) = J_\theta(g)\, \lambda_H$, is a right Haar measure on $G \ltimes H$.

Remark. $(h \circ \theta(g_0)^*(\lambda_H))(A) = \theta(g_0)^*(\lambda_H)(h^{-1}A)$

$$= \lambda_H(\theta(g_0)^{-1}(h^{-1}A)) = \lambda_H(\theta(g_0)^{-1}(h)^{-1}\, \theta(g_0)^{-1}(A))$$

$$= \lambda_H(\theta(g_0)^{-1}A) = \theta(g_0)^*(\lambda_H)(A)$$

$\Rightarrow \exists J_\Theta(g_0) \in \mathbb{R}^+$ s.t. $\Theta(g_0)^*(\lambda_H) = J_\Theta(g_0)\ \lambda_H$

One can easily see that $J_\Theta : G \to \mathbb{R}^+$ is a continuous group homomorphism.

__Pf__. Recall $(g_1, h_1)\cdot(g_2, h_2) = (g_1 g_2, \Theta(g_2^{-1})(h_1) h_2)$.

[You can remember this using the fact that $\Theta(g)(h)$ is supposed to be turned into conjugation:

$$\Theta(g)(h) = g h g^{-1}.$$

$$\Rightarrow g_1 \cdot h_1 \cdot g_2 \cdot h_2 = g_1 g_2 (g_2^{-1} h_1 g_2) h_2$$

$$= g_1 g_2\ \Theta(g_2^{-1})(h_1) h_2 .\ ]$$

Let $\mu = \lambda_G \times \lambda_H$. So $\forall f \in C_c(G \ltimes H)$

$$((g_0, h_0)\cdot\mu)(f) = \int_G \int_H f(\underbrace{(g_0, h_0)^{-1}}_{(g_0^{-1},\ \Theta(g_0)(h_0^{-1}))}(g, h))\, dh\, dg$$

$$= \int_G \int_H f(g_0^{-1} g, \Theta(g^{-1} g_0)(h_0^{-1}) h)\, dh\ dg$$

$$\stackrel{\boxed{dh \text{ left-invar.}}}{=} \int_G \int_H f(g_0^{-1} g, h')\, dh'\ dg$$

Fubini

$$\overset{\downarrow}{=} \int_H \int_G f(g_0^{-1}g, h)\, dg\, dh$$

dg left invar Fubini

$$\overset{\downarrow}{=} \mu(f).$$

Let $\nu(f) = \int_G \int_H f(g,h)\, J_\Theta(g)^{-1}\, dh\, dg$.

$$(\nu \cdot (g_0,h_0))(f) = \int_G \int_H f(gg_0, \Theta(g_0^{-1})(h)h_0) J_\Theta(g)^{-1}\, dh\, dg$$

$$= \int_G \int_H f(gg_0, \Theta(g_0^{-1})(h\, \Theta(g_0)(h_0))) J_\Theta(g)^{-1}\, dh\, dg$$

dh right invariant

$$\overset{\downarrow}{=} \int_G \int_H f(gg_0, \underbrace{\Theta(g_0^{-1})(h)}_{h'}) J_\Theta(g)^{-1}\, dh\, dg$$

$h = \Theta(g_0)(h')$

$d(\Theta(g_0)\, h') = J_\Theta(g_0^{-1})\, dh'$

$$\overset{\downarrow}{=} \int_G \int_H f(gg_0, h')\; J_\Theta(g)^{-1} \quad J_\Theta(g_0)^{-1}\, dh\, dg$$

Fubini

$J_\Theta : G \to \mathbb{R}^+$ homomorphism

$$\overset{\downarrow}{=} \int_H \int_G f(gg_0, h)\; J_\Theta(gg_0)^{-1}\, dg\, dh$$

dg right invariant Fubini

$$\overset{\downarrow}{=} \nu(f).$$

∎

Corollary Let $B = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{bmatrix} \;\middle|\; \begin{array}{l} a_{ij} \in \mathbb{R}, \\ a_{ii} \in \mathbb{R}^+ \end{array} \right\}$.

Then (0) $B \simeq A \ltimes N$ where A and N are as before

and $(\theta(a)(n))_{ij} = (ana^{-1})_{ij} = a_i a_j^{-1} n_{ij}$,

where $a = (a_1, \dots, a_n) \in \mathbb{R}^+ \times \cdots \times \mathbb{R}^+$.

(1) $da\,dn$ is a left Haar measure.

(2) $\rho(a)\,da\,dn$, where $\rho(a) = \prod_{i<j} (a_i a_j^{-1})$,

is a right Haar measure.

Pf. The only thing that we need to check is

$$J_\theta(a) = \rho(a)^{-1}.$$

$$(\theta(a)^* \lambda_N)(f) = \int_{\mathbb{R}^{\frac{n(n-1)}{2}}} f(\theta(a)^{-1} n(\vec{x}))\, d\vec{x}$$

$$= \int_{\mathbb{R}^{\frac{n(n-1)}{2}}} f(T(\vec{x}))\, d\vec{x}$$

$T(x_{ij}) = (a_i^{-1} a_j x_{ij})$

$$= \rho(a)^{-1} \int_{\mathbb{R}^{\frac{n(n-1)}{2}}} f(\vec{x})\, d\vec{x}$$

$$= \rho(a)^{-1} \lambda_N(f).$$ ∎

<u>Remark</u>. In particular, B is NOT unimodular.

<u>Theorem</u> Let G_1 and G_2 be two closed subgroups of G.

Suppose (1) G_1G_2 is an open subset of G and

$\lambda_G(G \setminus G_1G_2) = 0$.

(2) $H := G_1 \cap G_2$ is a compact subgroup.

Then

$$\int_G f(g)\, d\lambda_G(g) = \int_{G_1}\int_{G_2} f(g_1 g_2) \frac{\Delta_{G_2}(g_2)}{\Delta_G(g_2)}\, d\lambda_{G_2}(g_2)\, d\lambda_{G_1}(g_1)$$

I am leaving this as an exercise with a list of hints.

<u>Corollary</u> $dg = \rho(a)\, da\, dn\, dk$ is a Haar measure of $SL_n(\mathbb{R})$.

<u>Pf of corollary</u> By Gram-Schmidt

$$\underbrace{SO_n(\mathbb{R})}_{K} \times B \longrightarrow SL_n(\mathbb{R})$$

B: upper triang.

$$(k, b) \longmapsto kb$$

is onto; $K \cap B$ is finite; $B \simeq AKN$; Hence

we get the claim using the previous examples and theorem. ∎

Next we find a G-invariant measure on G/H whenever

it is possible.

Lemma. Let $I: C_c(G) \longrightarrow C_c(G/H)$,

$$(I(f))(gH) := \int_H f(gh)\, d\lambda_H(h).$$

where $H \leq G$ is a closed subgroup.
Then I is a well-defined onto map.

Pf. Since $f \in C_c(G)$, $L_{g^{-1}}(f)\, \mathbb{1}_H \in C_c(H)$. So the

integral exists. $\forall\, h_0 \in H$, $g \in G$ we have

$$\int_H f(gh_0h)\, d\lambda_H(h) = \int_H f(gh')\, d\lambda_H(h_0^{-1}h')$$

$$= \int_H f(gh')\, d\lambda_H(h')$$

$\Rightarrow$ $I(f)$ is a function on G/H.

Since f is compactly supported, its uniformly continuous $\Rightarrow$

$\forall \varepsilon>0$, $\exists\, U$ a nbhd of identity s.t. $\forall g\in G$ and $u\in U$

$$|f(gu)-f(g)|<\varepsilon \quad \text{and} \quad |f(ug)-f(g)|<\varepsilon.$$

$$\Rightarrow \quad I(f)(ugH)=\int_H f(ugh)\, d\lambda_H(h)$$

$$\leq \int_{H\cap (ug)^{-1}\mathrm{Supp}(f)} f(gh)+O(\varepsilon)\ d\lambda_H(h)$$

$$= I(f)(gH) + O(\varepsilon)\ \lambda_H(H\cap g^{-1}U\,\mathrm{supp}(f))$$

$\Rightarrow$ $I(f)$ is continuous.

$gH\notin \mathrm{Supp}(f)H \Rightarrow \forall h\in H,\ gh\notin \mathrm{Supp}(f)$

$$\Rightarrow I(f)(gH)=0.$$

$$\Rightarrow I(f) \text{ is compactly supported.}$$

Now let $\bar{f}\in C_c(G/H)$.

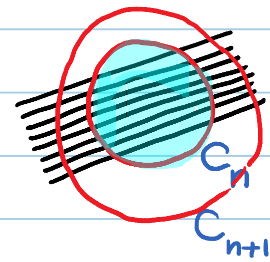
Since G is σ-compact, $\exists$ a sequence $\{C_n\}$ of nbhd of e s.t. $\overline{C_n}$ is compact and $\overline{C_n}\subseteq C_{n+1}$.

$\Rightarrow$ $\mathrm{Supp}(f)\subseteq \pi(C_n)$ for some n.

Let $\phi\in C_c(G)$ s.t. $\phi|_{\overline{C_n}}=1$ and $\phi|_{G\setminus C_{n+1}}=0$,

$0 \le \phi \le 1$.

Let $\tilde{f}(g) := \phi(g)\,\overline{f}(gH)$.



$$\Rightarrow I(\tilde{f})(gH) = \int_H \tilde{f}(gh)\, d\lambda_H(h)$$

$$= \overline{f}(gH) \int_H \phi(gh)\, d\lambda_H(h)$$

$$= I(\phi)(gH)\, \overline{f}(gH).$$

$\Rightarrow I(\tilde{f}) = I(\phi)\,\overline{f}$ and

$I(\phi)(gH) \neq 0$ if $\overline{f}(gH) \neq 0$.

Let $$f(g) := \begin{cases} \phi(g)\, \dfrac{\overline{f}(gH)}{I(\phi)(gH)} & gH \in \operatorname{Supp}(\overline{f}) \\ 0 & gH \notin \operatorname{Supp}(\overline{f}) \end{cases}$$

Notice that $\forall gH \in \pi(\overline{C}_n)$, $I(\phi)(gH) \neq 0$ and

$\overline{f}(gH) = 0$ if $gH \notin \pi(C_n) \Rightarrow f \in C_c(G)$ and

$I(f) = \overline{f}$. ∎

<u>Lemma</u>. Suppose $\Delta_G|_H = \Delta_H$. Then

$$f \in \ker I \Rightarrow \int_G f(g)\, d\lambda_G(g) = 0$$

$f \in \operatorname{Ker} I \Rightarrow \int_G f(g)\, d\lambda_G(g) = 0$

Pf. $I(f)(gH) = \int_H f(gh)\, d\lambda_H(h) = 0 \quad \forall g \in G.$

$$\Rightarrow \forall \phi \in C_c(G), \quad \int_G \int_H f(gh)\phi(g)\, d\lambda_H(h)\, d\lambda_G(g) = 0$$

$\Rightarrow F(g,h) := f(gh)\phi(g) \in C_c(G \times H) \rightsquigarrow$ by Fubini

$$0 = \int_H \int_G f(gh)\,\phi(g)\, d\lambda_G(g) \quad d\lambda_H(h)$$

$$= \int_H \int_G f(g')\,\phi(g'h^{-1})\, d\lambda_G(g'h^{-1}) \quad d\lambda_H(h)$$

$$= \int_H \int_G f(g)\,\phi(gh^{-1})\,\Delta_G(h^{-1})\, d\lambda_G(g) \quad d\lambda_H(h)$$

$$= \int_G f(g) \int_H \phi(gh^{-1})\,\Delta_G(h^{-1})\, d\lambda_H(h) \quad d\lambda_G(g)$$

Fubini

$$= \int_G f(g) \int_H \phi(gh')\,\Delta_G(h')\, d\lambda_H(h'^{-1})\, d\lambda_G(g)$$

$$= \int_G f(g) \left(\int_H \phi(gh)\, d\lambda_H(h) \right) d\lambda_G(g)$$

$$= \int_G f(g)\, I(\phi)(gH)\, d\lambda_G(g).$$

We can choose ϕ so that $I(\phi)\big|_{\pi(\mathrm{Supp}(f))} = 1.$

We can choose ϕ so that $\mathbb{1}(\phi)\Big|_{\pi(\mathrm{Supp}(f))} = 1$.

$$\Rightarrow \quad 0 = \int_G f(g)\, d\lambda_G(g). \qquad \blacksquare$$

<u>Lemma</u>. Suppose $\Delta_G\big|_H = \Delta_H$. Let

$$\ell : C_c(G/H) \longrightarrow \mathbb{C}, \quad \ell(I(f)) := \int_G f(g)\, d\lambda_G(g)$$

for any $f \in C_c(G)$. Then ℓ is a well-defined positive function which is G-invariant.

<u>Pf</u>. $I(f_1) = I(f_2) \Rightarrow I(f_1 - f_2) = 0 \Rightarrow \underbrace{\int_G f_1(g) - f_2(g)\, d\lambda_G(g)}_{=0}$

$$\Rightarrow \int_G f_1(g)\, d\lambda_G(g) = \int_G f_2(g)\, d\lambda_G(g)$$ so it is well-defined.

Going through the proof of surjectivity of I, we see that, if $\underline{\bar{f} \geq 0}$, then $\exists\, f \geq 0$ s.t. $I(f) = \bar{f}$.

So $\underline{\ell}$ is a positive function.

$$L_g(\ell)(I(f)) = \ell(L_{g^{-1}}(I(f)))$$
$$= \ell(I(L_{g^{-1}}(f)))$$
$$= \int_G L_{g^{-1}}(f)(g')\, d\lambda_G(g')$$

$$= \int_G L_{g^{-1}}(f)(g')\, d\lambda_G(g')$$

$$= \int_G f(g')\, d\lambda_G(g')$$

$$= \ell(I(f)). \qquad \blacksquare$$

<u>Thm</u>. Suppose $\Delta_G|_H = \Delta_H$. Then $\exists!$ (up to a scalar) Radon measure ν on G/H which is G-invariant and $\forall f \in C_c(G)$, $\int_G f(g)\, d\lambda_G(g) = \int_{G/H} \int_H f(gh)\, d\lambda_H(h)\, d\nu$.

<u>Pf.</u> By the previous Lemma and Riesz Representation Thm, we get the desired result. $\blacksquare$

<u>Thm</u> If G/H has a G-invariant Radon measure, then

$$\Delta_G|_H = \Delta_H.$$

<u>Pf.</u> Let $\mu(f) := \int_{G/H} I(f)(gH)\, d\nu(gH)$. Then clearly $\mu(f)$ is a positive G-invariant function on $C_c(G)$. So by the uniqueness of Haar measure we have

$$\int_G f(g)\, d\lambda_G(g) = \int_{G/H} \int_H f(gh)\, d\lambda_H(h)\, d\nu$$

$$\Rightarrow \int_G f(gh')\, d\lambda_G(g) = \int_{G/H} \int_H f(ghh')\, d\lambda_H(h)\, d\nu$$

$$\Rightarrow \Delta_G(h')^{-1} \int_G f(g)\, d\lambda_G(g) = \Delta_H(h')^{-1} \int_{G/H} \int_H f(gh)\, d\lambda_H(h)\, d\nu$$

$$= \Delta_H(h')^{-1} \int_G f(g)\, d\lambda_G(g)$$

$$\Rightarrow \Delta_G\big|_H = \Delta_H . \qquad \blacksquare$$

Lemma. $H_1 \subseteq H_2 \subseteq G$; If G/H_1 has a G-invariant finite measure, then so do G/H_2 and H_1/H_2; and vice versa. Moreover we have

$$\int_{G/H_1} f(gH_1)\, d\lambda_{G/H_1} = \int_{G/H_2} \int_{H_2/H_1} f(gh_2H_1)\, d\lambda_{H_2/H_1}(h_2H_1)\, d\lambda_{G/H_2}(gH_2)$$

Outline of proof. $\pi: G/H_1 \longrightarrow G/H_2 \Rightarrow \pi_*(\nu)$ is an G-invariant finite measure:

$$\pi_*(\nu)(f) = \int_{G/H_1} f(gH_2)\, d\lambda_{G/H_1}$$

is well-defined as λ_{G/H_1} is a finite measure.

$$\Rightarrow \Delta_G\big|_{H_1} = \Delta_{H_1} \text{ and } \Delta_G\big|_{H_2} = \Delta_{H_2} \Rightarrow \Delta_{H_2}\big|_{H_1} = \Delta_{H_1}$$

$\Rightarrow \Delta_G|_{H_1} = \Delta_{H_1}$ and $\Delta_G|_{H_2} = \Delta_{H_2} \Rightarrow \Delta_{H_2}|_{H_1} = \Delta_{H_1}$

$\Rightarrow \lambda_{H_2/H_1}$ exists.

The rest can be deduce considering

$$f \longmapsto \int_{G/H_2} \int_{H_2/H_1} d\lambda_{H_2/H_1}(f(gh_2H))\, d\lambda_{G/H_2}(gH_2).$$

And using Fubini. $\square$

Important Remark. Mackey observed that $\exists$ a Borel section $s: G/H \to G$. Let $\theta: G/H \times H \to G$, $\theta([g],h) = s([g])h$, and $\mu = \theta_*(\lambda_{G/H} \times \lambda_H)$ (assuming $\Delta_G|_H = \Delta_H$).

Therefore $\forall f \in C_c(G)$,

$$\int_G f(g)\, d\mu = \int_{G/H} \int_H f(s[g]h)\, d\lambda_H\, d\lambda_{G/H}$$

$$= \int_{G/H} I(f)(\pi(s[g]))\, d\lambda_{G/H}$$

$$= \int_{G/H} I(f)([g])\, d\lambda_{G/H} = \int_G f(g)\, d\lambda_G$$

$$\Rightarrow \lambda_G = \theta_*(\lambda_{G/H} \times \lambda_H)$$

$\Rightarrow$ By Fubini, $\underline{\forall f \in L^1(G)}$, $I(f) := \int_H f(gh)\, dh \in L^1(G/H)$

and $\int_G f(g)\, d\lambda_G = \int_{G/H} \int_H f(gh)\, d\lambda_H\, d\lambda_{G/H}$.

and $\int_G f(g)\, d\lambda_G = \int_{G/H} \int_H f(gh)\, d\lambda_H\, d\lambda_{G/H}$.

Def. A subgroup Γ of G is said to be a lattice in G

if (1) Γ is a discrete subgroup.

(2) G/Γ has a finite G-invariant Radon measure.

Lemma If G has a lattice, then G is unimodular.

Pf. Since G/Γ has a G-invariant measure,

$\Delta_G|_\Gamma = \Delta_\Gamma$. Since Γ is discrete, $\Delta_\Gamma = 1$.

$\Rightarrow \Gamma \subseteq \ker \Delta_G = N \Rightarrow G/N \hookrightarrow \mathbb{R}^+$

is a subgroup with finite G-invariant measure

$\Rightarrow G/N$ is a compact subgroup of $\mathbb{R}^+ \Rightarrow G = N \Rightarrow \Delta_G = 1$. ■

Corollary $B = A \ltimes N$ upper-triangular matrices has no lattice.

Lemma. Suppose G is unimodular, $\Gamma \subseteq G$ is a discrete subgroup. If $\exists\, \mathcal{F} \subseteq G$ s.t.

(0) $\mathcal{F}$: Borel subset,

(0) $\mathcal{F}$: Borel subset,

(1) $\lambda_G(\mathcal{F}) < \infty$;

(2) $G = \bigcup_{\gamma \in \Gamma} \mathcal{F}\gamma$,

then Γ is a lattice in G.

<u>Pf</u>. Since G and Γ are unimodular, $\lambda_{G/\Gamma}$ exists.

Since $\mathcal{F}$ is a Borel set and $\lambda_G(\mathcal{F}) < \infty$, $\mathbb{1}_{\mathcal{F}} \in L^1(\lambda_G)$.

So by (Weil's formula) the above formula

$$\lambda_G(\mathcal{F}) = \int_G \mathbb{1}_{\mathcal{F}} \, d\lambda_G = \int_{G/\Gamma} I(\mathbb{1}_{\mathcal{F}})([g]) \, d\lambda_{G/\Gamma}.$$

$$I(\mathbb{1}_{\mathcal{F}})([g]) = |\{\gamma \in \Gamma \mid g\gamma \in \mathcal{F}\}| \geq 1 \text{ as}$$

$$G = \bigcup_{\gamma \in \Gamma} \mathcal{F}\gamma.$$

$\Rightarrow \quad \infty > \lambda_G(\mathcal{F}) \geq \lambda_{G/\Gamma}(G/\Gamma) \Rightarrow \Gamma$ is a lattice. ∎