Proof of Oppenheim conjecture

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Let $Q_0(x,y,z) = y^2 + xz = -\det\begin{bmatrix} y & x \\ z & -y \end{bmatrix}$.

Identifying $\mathfrak{sl}_2(\mathbb{R})$ with $\mathbb{R}^3$, it is clear that $\mathrm{Ad}(SL_2(\mathbb{R}))$ preserves Q_0. So we get a homomorphism from $SL_2(\mathbb{R})$ to $SO_{Q_0}(\mathbb{R})$. Let's see where the upper-triangular matrices are mapped to in $SL_3(\mathbb{R})$.

$$\mathrm{Ad}\left(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} y & x \\ z & -y \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y & x \\ z & -y \end{bmatrix}\begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} y+tz & x-ty \\ z & -y \end{bmatrix}\begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} y+tz & -ty-t^2z+x-ty \\ z & -tz-y \end{bmatrix}$$

$$\mapsto \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x-2ty-t^2z \\ y+tz \\ z \end{bmatrix}$$

$$\leadsto \begin{bmatrix} 1 & -2t & -t^2 \\ & 1 & t \\ & & 1 \end{bmatrix} \in SO_{Q_0}(\mathbb{R})$$

$$\mathrm{Ad}\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix}\left(\begin{bmatrix} y & x \\ z & -y \end{bmatrix}\right) = \begin{bmatrix} y & a^2x \\ a^{-2}z & -y \end{bmatrix}$$

$$\mapsto \begin{bmatrix} a^2 & & \\ & 1 & \\ & & a^{-2} \end{bmatrix} \in SO_{Q_0}(\mathbb{R}).$$

Since the adjoint action is irreducible, $C_{M_3(\mathbb{R})}(SO_{Q_0}(\mathbb{R})^\circ) = \mathbb{R}I$.

Proposition If $Q(\vec{v}) := Q_0(g\vec{v})$ is "irrational", then $HgΓ$ is NOT closed in $G/Γ$.

Pf. Suppose $\pi(Hg) \subseteq G/Γ$ is closed where $\pi : G \to G/Γ$.

Notice that $SO_Q(\mathbb{R}) = g^{-1} SO_{Q_0}(\mathbb{R})\, g$:

$$\begin{aligned} Q(g^{-1}hg\ \vec{v}) &= Q_0(g \cdot g^{-1} h g \vec{v}) \\ &= Q_0(hg\vec{v}) \\ &= Q_0(g\vec{v}) \\ &= Q(\vec{v}). \end{aligned}$$

So $\pi(SO_Q(\mathbb{R})) = g^{-1}\pi(Hg)$ is closed in $G/Γ$.

Let $\Delta := Γ \cap SO_Q(\mathbb{R})$. Since $\pi(SO_Q(\mathbb{R}))$ is closed, π induces a homeomorphism

$$SO_Q(\mathbb{R})/\Delta \longrightarrow \pi(SO_Q(\mathbb{R})).$$

[**Proposition**. G, X : locally compact, second countable, Hausdorff.

$G \curvearrowright X$ continuously and transitively. Then for any $x \in X$

$$\begin{aligned} G/G_x &\longrightarrow X \\ gG_x &\longmapsto g \cdot x \end{aligned}$$

is a homeomorphism.

Pf. Clearly, it is a continuous bijection. Suppose $g_i \cdot x \to g \cdot x$ in X. We'd like to show $g_i G_x \to g G_x$. W.L.O.G we can assume that

$g=e$. Suppose to the contrary that $g_i G_x \not\to G_x$. So $\exists$ a nbhd $\mathcal{O}$ of e s.t. $g_i G_x \cap \mathcal{O} = \emptyset$ for $i \gg 1$. Therefore $g_i \notin \mathcal{O} G_x$, which implies that $g_i \cdot x \notin \mathcal{O} \cdot x$. So to get a contradiction it is enough to show that $\mathcal{O}.x$ contains a nbhd of x. Let $\{g_i'\}$ be a countable dense subset of G. Let $\mathcal{O}'$ be a nbhd of e s.t. $\overline{\mathcal{O}'} \subseteq \mathcal{O}$ is compact and $\mathcal{O}'^{-1}\mathcal{O}' \subseteq \mathcal{O}$.

$\Rightarrow G = \bigcup g_i' \overline{\mathcal{O}'} \Rightarrow X = \bigcup g_i' \overline{\mathcal{O}'}.x$ is a countable union of compact subsets. By Baire Category, $\exists i$ s.t. $g_i' \overline{\mathcal{O}'}.x$ has a non-empty interior, which implies that $\overline{\mathcal{O}'}.x$ has a non-empty interior. Hence $\exists g \in \overline{\mathcal{O}'}$ s.t. $g^{-1}\overline{\mathcal{O}'}.x$ contains a nbhd of $x \Rightarrow \overline{\mathcal{O}'}^{-1}\overline{\mathcal{O}'}.x \subseteq \mathcal{O}.x$ contains a nbhd of x. ■

<u>Remark</u>. A more general statement is true: G, X: locally comp., s.c., Haus.

$G \curvearrowright X$. all the orbits are locally closed

$$\Updownarrow$$

$$G/G_x \simeq G.x$$

(See Zimmer, *Ergodic theory and semisimple groups.*)]

The general idea is that Δ should be Zariski-dense which implies that SO_Q is defined over $\mathbb{Q}$, and that should imply that Q is rational.

Here is what we actually need: if a quadratic form is preserved by Δ, then it is a multiple of Q. (A kind of Zariski-density)

Let $V = \{ S \in M_n(\mathbb{R}) \mid S = S^t,\ \forall \lambda \in \Delta,\ \lambda^t S \lambda = S \}$.

So V is a vector space defined by linear relations with coeff. in $\mathbb{Q}$. The symmetric matrix of Q is in V. Now suppose

$Q'(\vec{v}) = \vec{v}^{\,t} S' \vec{v}$ for some $S' \in V$ (it might be degenerate)

So $\forall \lambda \in \Delta$, $Q'(\lambda \vec{v}) = Q'(\vec{v})$. $\circledast$

Let $f_{\vec{v}_0} : SO_Q(\mathbb{R}) \longrightarrow \mathbb{R}$, $f_{\vec{v}_0}(g) = Q'(g^{-1}\vec{v}_0)$.

By $\circledast$ $f_{\vec{v}_0}$ factors through $SO_Q(\mathbb{R})/\Delta$ which is homeo. to $\pi(SO_Q(\mathbb{R}))$.

Let $\{u_t\}$ be a unipotent flow of $SO_Q(\mathbb{R})$. Then

$t \longmapsto f_{\vec{v}_0}(u_t)$ is a polynomial.

On the other hand, $\{\pi(u_t)\}$ returns to a compact subset of $\pi(SO_Q(\mathbb{R}))$ infinitely often. So the polynomial $f_{\vec{v}_0}(u_t)$ does NOT go to $\pm$ infinity as $t \to \infty$. Therefore it is constant.

$\Rightarrow \forall \vec{v}_0$ and unip. flow $u_t \subseteq SO_Q(\mathbb{R})$ we have

$$Q'(u_t \vec{v}_0) = Q'(\vec{v}_0).$$

$\Rightarrow Q'$ is invariant under $g_Q \mathrm{Ad}(SL_2(\mathbb{R})) g_Q^{-1}$ under the above identification of $\mathfrak{sl}_2(\mathbb{R})$ and $\mathbb{R}^3$. ($SL_2(\mathbb{R})$ is gen. by its unipotent flows.)

$\Rightarrow \forall g \in SO(Q)$, $g^t S g = S$ and $g^t S' g = S'$

where $v^t S v = Q(v)$

$\Rightarrow S^{-1} S' g = S^{-1} (g^t)^{-1} S' = (g^t S)^{-1} S' = (S g^{-1})^{-1} S' = g S^{-1} S'$

$\Rightarrow \quad S^{-1}S' \in C_{M_3(\mathbb{R})}(SO(Q)) = \mathbb{R}\cdot I.$

$\Rightarrow \quad S' \in \mathbb{R}\cdot S.$

$\Rightarrow \dim_{\mathbb{R}} V = 1.$

Since V is given by rational linear equations, it has a rational vector S_0. So S is (proportional to) a rational quadratic form. ∎

Margulis Proved the following:

1 Thm. Let $H = SO_{Q_0}(\mathbb{R})^{\circ}$. Any bounded H-orbit in $SL_3(\mathbb{R})/SL_3(\mathbb{Z})$ is compact.

The above theorem implies, $\forall \varepsilon > 0$, $\exists \vec{v} \in \mathbb{Z}^3$ s.t. $|Q(v)| < \varepsilon$ where Q is as above.
$\vec{v} \neq 0$

If not, $|Q(\mathbb{Z}^3 \setminus \{0\})| \geq \varepsilon_0$ for some $\varepsilon_0 > 0$. Hence

$Q_0(g(\mathbb{Z}^3) \setminus \{0\}) \geq \varepsilon_0 \Rightarrow \forall h \in H,\ Q_0(hg(\mathbb{Z}^3) \setminus \{0\}) \geq \varepsilon_0$

$\Rightarrow \delta(hg\mathbb{Z}^3) \gg 1 \Rightarrow Hg\mathbb{Z}^3$ is bounded in $\Omega^{(1)}(\mathbb{R}^3)$

by Mahler's compactness criteria.

$\Rightarrow$ by the above theorem $\pi(Hg)$ is compact

$\Rightarrow$ By the above proposition Q cannot be irrational, which is a contradiction. ∎

2 Dani-Margulis Under the same assumptions,

$$\overline{Q(\text{primitive integral vectors})} = \mathbb{R}.$$

We prove the following stronger dynamical theorem, which implies the above strong form of Oppenheim's Conjecture.

③ <u>Thm (Dani–Margulis)</u> Hx is NOT close in $\Omega^{(1)}(\mathbb{R}^3) \Rightarrow \exists y_0$ s.t. either $V^{\geq} y_0 \subseteq \overline{Hx}$ or $V^{\leq} y_0 \subseteq x$, where

$V = \left\{ \begin{bmatrix} 1 & & t \\ & 1 & \\ & & 1 \end{bmatrix} \middle|\ t \in \mathbb{R} \right\}$, $V^{\geq} = \{v(t) \mid t \geq 0\}$, $V^{\leq} = \{v(t) \mid t \leq 0\}$,

and $H = SO(Q_0)$.

<u>Thm ③ $\Rightarrow$ ②</u>

$$Q_0\left(v(t)\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = y^2 + (x+tz)z = y^2 + xz + tz^2.$$
$$= Q_0(x,y,z) + tz^2.$$

Let ω_0 be a primitive vector of y_0 with a non-zero third component.

$\Rightarrow Q_0(V^{\geq}\omega_0) \supseteq [Q_0(\omega_0), \infty)$. For $a<0$,

consider $\mathcal{O}_a = \{ w \in \mathbb{R}^3 \mid Q_0(w) < a,\ w_3 \neq 0 \}$. So $\mathcal{O}_a$ is a non-empty open subset of $\mathbb{R}^3$ and $\forall t \in \mathbb{R}^{\geq 1}$, $t\,\mathcal{O}_a \subseteq \mathcal{O}_a$.

<u>Claim</u>. $\Lambda \in \Omega^{(1)}(\mathbb{R}^3) \Rightarrow \Lambda \cap \mathcal{O}_a$ has a primitive vector.

<u>Pf of claim</u>. $\Lambda = g\mathbb{Z}^3 \Rightarrow$ we need to show $\mathbb{Z}^3 \cap g^{-1}\mathcal{O}_a$ has a primitive vector. And $g^{-1}\mathcal{O}_a$ has the following properties:

① Non-empty open subset of $\mathbb{R}^3$.

② $\forall t \in \mathbb{R}^{\geq 1}$, $t\,\mathcal{O} \subseteq \mathcal{O}$.

Since $\mathcal{O}$ is open, $\exists$ two linearly independent rational vectors w_1 and $w_2 \in \mathcal{O}$ s.t. the segment $t w_1 + (1-t) w_2$ is also in $\mathcal{O}$ ($\cdot s t \le 1$). After rescaling, $\exists\, \omega_1, \omega_2 \in \mathcal{O} \cap \mathbb{Z}^{\smile}$ that are linearly independent and

$\forall\, t_1, t_2 \in \mathbb{R}^{\geq 0}$, $t_1 + t_2 \geq 1 \Rightarrow t_1 \omega_1 + t_2 \omega_2 \in \mathcal{O}$.

Applying $\gamma \in SL_3(\mathbb{Z})$ to $\mathcal{O}$, we can further assume that $\vec{\omega}_1 = m \vec{e}_1$. Suppose the second component of ω_2 is $m_2 \neq 0$. Let $p \gg 1$ prime. Then

$$\frac{p - m_1}{m}(m\vec{e}_1) + (m_1, m_2, m_3) = (p, m_2, m_3) \in \mathcal{O}$$

and is primitive. □

So $Q_0(\omega_0)$ can be arbitrarily small as ω_0 varies among primitive vectors of y_0.

$V^2 y_0 \subseteq \overline{Hx} \Rightarrow$

$\forall\, r \in \mathbb{R}$, $\exists\, t \in \mathbb{R}$, $\vec{\omega}$ primitive in y_0 s.t. $Q_0(v(t)\omega) = r$

$\Rightarrow \exists\, h_i \in H$, ω_i primitive in x s.t.

$$h_i \omega_i \longrightarrow u(t)\omega \Rightarrow Q_0(h_i \omega_i) \longrightarrow r$$

$$\Rightarrow Q_0(\omega_i) \longrightarrow r \Rightarrow \overline{Q(\text{primitive in } \mathbb{Z}^3)} = \mathbb{R}. \quad \blacksquare$$

Thm ② $\Rightarrow$ ① Hx bounded not compact $\Rightarrow Hx$ not closed

$\Rightarrow \overline{Q_0(x)} = \mathbb{R} \Rightarrow \forall\, \varepsilon > 0, \exists\, h \in H$, $\delta(hx) < \varepsilon$

$\Rightarrow Hx$ is NOT bounded. ■

. Hx is NOT closed, $\overline{Hx} \supsetneq Hx$ is H-invariant.

$H^\circ \simeq \mathrm{Ad}(SL_2(\mathbb{R}))$ if we identify $\mathbb{R}^3$ with $\mathfrak{sl}_2(\mathbb{R})$ using $\left\{ \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ basis and the fact that $\det(\mathrm{Ad}(g) X) = \det(X)$. [In the above basis, we have

$$\det\left(x\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} + y\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + z\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \det\begin{bmatrix} y & -2x \\ z & -y \end{bmatrix} = -y^2 + 2xz = Q_0(x,y,z)$$

(The associated symmetric matrix is $\begin{bmatrix} & & 1 \\ & -1 & \\ 1 & & \end{bmatrix}$.).]

- For both measure rigidity of unipotent flows and the topological approach of understand orbit closures of unipotent flows, one needs to come up with a process of producing "<u>extra invariance</u>". In the measure rigidity case, it is more clear what this means. In the topological case, as we have learned in $\times 2\ \times 3$ theorem, (due Hedlund in the $SL_2(\mathbb{R})$-case) one needs to work with minimal subsets. The following general statement guides us to the direction of finding new invariance:

<u>Proposition</u>. G: locally compact, second countable, Hausdorff;

$U \subseteq H \subseteq G$ closed subgroups;

$G \curvearrowright \Omega$ continuously; l.c., s.c., Hausdorff;

$X \subseteq \Omega$: H-invariant, closed,

$Y \subseteq X$: compact, U-minimal

$M \subseteq G$; $\forall m \in M$, $mY \cap X \neq \emptyset$

$$\Downarrow$$

$\forall g \in N_G(U) \cap \overline{HMU}$, $gY \subseteq X$.

U can be replaced with $\{g \in G \mid gY = g\}$

$$\forall g \in N_G(U) \cap \overline{HMU}, \quad gY \subseteq X. \quad \{g\in G \mid gY = g\}$$

In particular, if $U \subseteq B$, Y is B-invariant, and $mY \cap Y \neq \emptyset$, then $\forall g \in N_G(U) \cap \overline{BMB}$, $gY = Y$. $(X = Y,\ H = B)$

Pf is short. Leave it as an exercise. $\square$

Where are we?

$Hx \subsetneq \overline{Hx}$ with a general conjecture that $\overline{Hx}$ should be a homogenous space. So $\{g \in G \mid gx \in \overline{Hx}\}$ should be a closed subgroup which contains H. Hence by maximality of H (why?) this should be the entire $G = SL_3(\mathbb{R})$. Here we only want to show $\overline{Hx}$ contain an orbit of $V^{>}$ or $V^{<}$ (not necessarily the x orbit.)

The main property of unipotent flows that help us to get "extra invari." is the following Lemma which is about limits of parts of U-orbits under a linear action. Before stating the lemma, let me say the relevance of "new invariance" and linear actions. This is closer to the truth in the measure rigidity case. Say H is in "the group of invariance", and we would like to get something outside H. So we need to get something non-trivial in G/H. By Chevalley's theorem, if H

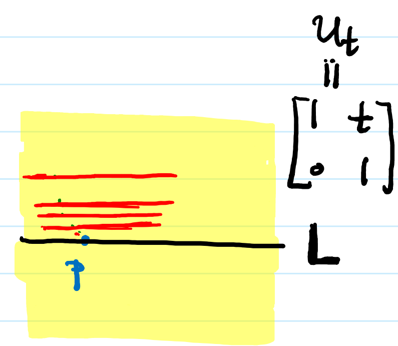
is an algebraic subgroup, G/H can be identified with a quasi-projective subvariety of $\mathbb{P}(\mathbb{V})$ via a linear action of G on $\mathbb{V}$ and a line that is fixed only by H. We are in a better situation if the quotient map has a "nice" section.

<u>Lemma</u> Suppose $\{u_t\}_{t\in\mathbb{R}} \subseteq GL(E)$ be a unipotent flow. Let $L := \{w \in E \mid u_t w = w \ \forall t\}$, $M \subseteq E \setminus L$, and $p \in L \cap \overline{UM}$.

Then $\exists$ a polynomial map $\varphi : \mathbb{R} \to L$ s.t.

① $\varphi(0) = p$, and non-constant

② $\mathrm{Im}(\varphi) \subseteq L \cap \overline{UM}$.



As before we identify $H^\circ = SO(Q_0)^\circ$ with $\mathrm{Ad}(SL_2(\mathbb{R}))$.

Let U be the image of $\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$, A be the image of $\begin{bmatrix} \alpha & \\ & \alpha^{-1} \end{bmatrix}$, and $B = AU \simeq A \ltimes U$. To be precise,

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y & -2x \\ z & -y \end{bmatrix}\begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} y+tz & -2x-ty \\ z & -y \end{bmatrix}\begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} y+tz & -ty-t^2z-2x-ty \\ z & -tz-y \end{bmatrix}$$

$$= z\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + (y+tz)\begin{bmatrix} 1 & \\ & -1 \end{bmatrix} + \left(+ty+x+\frac{t^2}{2}z\right)\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}.$$

So it is $\begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} =: u_t$.

Similarly $\begin{bmatrix} \alpha & \\ & \alpha^{-1} \end{bmatrix}\begin{bmatrix} y & -2x \\ z & -y \end{bmatrix}\begin{bmatrix} \alpha^{-1} & \\ & \alpha \end{bmatrix} = \begin{bmatrix} y & -2\alpha^2 x \\ \alpha^{-2} z & -y \end{bmatrix}$. So

$$A = \left\{ \begin{bmatrix} \alpha^{-2} & & \\ & 1 & \\ & & \alpha^{2} \end{bmatrix} \;\middle|\; \alpha \in \mathbb{R}^{+} \right\}.$$

So $\overline{Hx}$ is $\{u_t\}$-invariant. Hence by a theorem that we proved using quantitative non-divergence of unipotent flows, there is $Y \subseteq \overline{Bx} \subseteq \overline{Hx}$ which is $\{u_t\}$-minimal (i.e. any $\{u_t\}$-orbit is dense).

Claim 1. Y is compact

(Claim 1 is true for a flow which does NOT diverge.)

(For the weak form of Oppenheim conjecture, $\overline{Hx}$ is compact, and it is clear that Y exists and it is compact.)

Now starting from a point in $\overline{Bx}$ which very close to Y, we follow the unipotent flow. Since $\overline{Bx}$ is U-invariant, we will stay in $\overline{Bx}$.

Picture

$f(\varepsilon) \xrightarrow[\varepsilon \to 0]{} \infty$

For $t_\varepsilon \in [f(\varepsilon), C f(\varepsilon)]$, $d(u_t y, u_t y_\varepsilon) = O(1)$ and the most growth happens in the direction of fixed points of $\{u_t\}$.

We can choose t_ε so that $u_{t_\varepsilon} y_\varepsilon$ is in a compact set. So going to a subseq. $u_{t_{\varepsilon_i}} y_{\varepsilon_i} \longrightarrow y''$ and $u_{t_i} y \to y'$. And $y'' = v\, y'$ where the $\{u_t\}$-commutes with v. Having more control on $\{y_\varepsilon\}$'s gives us more information on v. Besides v can be chosen as a polynomial map as in the above lemma.

can be chosen as a polynomial map as in the above lemma.

For $y \in Y$, let $M_y := \{g \in G \mid gy \in \overline{Bx}\}$. So clearly

- $\forall m \in M_y$, $mY \cap X \neq \emptyset$ where $X = \overline{Bx}$.
- X is B-invariant
- Y is U-minimal

$\Rightarrow \forall g \in N_G(U) \cap \overline{HM_yU}$, $gY \subseteq \overline{Hx}$, and

$\forall g \in N_G(U) \cap \overline{BM_yU}$, $gY \subseteq \overline{Bx}$

(by the above proposition.)

. Let's find $N_G(U)$. (A general remark: $\boxed{\text{FYI}}$ for any unipotent group there is a parabolic subgroup P s.t. $U \subseteq R_u(P)$ and $N(U) \subseteq P$.)

$N_G(U)/C_G(U) \hookrightarrow \mathrm{Aut}(\mathbb{R}) \simeq \mathbb{R}^\times$ (as topological groups.)

$\Rightarrow N_G(U) = \left\{ \begin{bmatrix} \alpha & & \\ & 1 & \\ & & \alpha^{-1} \end{bmatrix} \mid \alpha \in \mathbb{R}^\times \right\} C_G(U)$.

$$\begin{bmatrix} 1 & t & t^2/2 \\ & 1 & t \\ & & 1 \end{bmatrix} = \exp\left(t \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) \Rightarrow C_G(U) = C_G\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$= \left\{ \begin{bmatrix} 1 & a & b \\ & 1 & a \\ & & 1 \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

$$= UV$$

where $V = \left\{ \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid t \in \mathbb{R} \right\}$.

So $N_G(U)^\circ = BV$.

And as we have seen above $v(t) = \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \notin H$. In fact,

$Q_0\left(v(t) \begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = Q_0(x,y,z) + 2tz^2$. So V gives us a perfect

$Q_0(w(t)\begin{bmatrix} x\\ y\\ z\end{bmatrix}) = Q_0(x,y,z) + 2tz^2$. So V gives us a perfect section for

$$H\backslash {}^{H}N_G(U) \longleftrightarrow {}_{H\cap N_G(U)}\backslash N_G(U) \longleftrightarrow V.$$

. In the spirit of Chevalley's theorem, we will give a rep'n of G and a vector whose stabilizer is H:

$\rho: G \to GL(E)$ where $E := \{S \in M_3(\mathbb{R}) \mid S = S^t\}$,

$\rho(g)(S) = (g^{-1})^t S (g^{-1})$. (it gives us a right G-action: $S \cdot g := \rho(g^{-1})(S)$.)

Let $S_0 = \begin{bmatrix} 0 & 0 & 1\\ 0 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}$. Then $\{g \in G \mid \rho(g)(S_0) = S_0\} = H$.

And the G-orbit of S_0 is the set of sgn $(-2,1)$ symmetric matrices which is closed. Hence

$$H\backslash G \simeq S_0 \cdot G$$

$$Hg \longmapsto g^t S_0 g \qquad \subseteq SL_3(\mathbb{R}) \cap \text{Symm. matrices}.$$

Now we will use the above lemma about orbits of unipotent flows for this linear action.

We start by understanding the fixed points of $\rho(U)$:

$$L := \{S \in M_3(\mathbb{R}) \mid S = S^t,\ \rho(\underbrace{\begin{bmatrix} 1 & t & t^2/2\\ & 1 & t\\ & & 1\end{bmatrix}}_{u(t)})(S) = S\}$$

<u>Claim 2</u>. $L = \mathrm{Span}(S_0, S_1)$ where $S_0 = \begin{bmatrix} & & 1\\ & -1 & \\ 1 & & \end{bmatrix}$ and $S_1 = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1\end{bmatrix}$.

And $L \cap SL_3(\mathbb{R}) = \{S_0 + tS_1 \mid t \in \mathbb{R}\} = S_0 \cdot V$.

To use the mentioned lemma we need to have a subset of $E_\rho \backslash L$.

Notice that $\quad S_0\cdot g \in L \Leftrightarrow S_0 \cdot g \in S_0 \cdot V$

$$\Leftrightarrow V g^{-1} \subseteq H$$

$$\Leftrightarrow g \in HV.$$

Suppose, for some $y_0 \in Y$ (a U_- minimal set that was chosen),

$M_{y_0} := \{g\in G \mid g\, y_0 \in \overline{Hx}\}$ contains a sequence $g_i \to e$ and

$g_i \in G\setminus HV$.

Under the above assumption, let $M := \{S_0 \cdot g_i\}$

Then $M \subseteq E_p \setminus L$, $S_0 \in L \cap \overline{M}$. So by the above lemma

$\exists$ a polynomial map $\phi: \mathbb{R} \to L$ s.t.

① $\phi(0) = S_0$ and non-constant

② $\phi(s) \in L \cap \overline{\rho(U)\{\rho(g_i^{-1})(S_0)\}_i} \subseteq L \cap \overline{\rho(U M_{y_0}^{-1})(S_0)}$

$$\subseteq \rho(V)(S_0) = S_0 \cdot V$$

Notice that $\quad S_0 \cdot v(t) = S_0 + 2t\, S_1$ (as computed above). So

there is a non-constant polynomial $p(s)$ s.t.

① $p(0) = 0$, ② $S_0 + 2p(s)S_1 = S_0 \cdot v(p(s)) \in \overline{S_0 \cdot (M_{y_0} U)}$

$\Rightarrow \quad v(p(s)) \in \overline{H M_{y_0} U} \cap N_G(U)$

$\Rightarrow \quad v(p(s))\, Y \subseteq \overline{Hx}$

Since $p(s)$ is a non-constant poly. and $p(0)=0$, either $\mathbb{R}^{\geq 0}$

or $\mathbb{R}^{\leq 0}$ is a subset of $\mathrm{Im}(p)$

$\Rightarrow \forall y \in Y$, $\quad$ either $V^{\geq} y \subseteq \overline{Hx}$ or $V^{\leq} y \subseteq \overline{Hx}$.

Suppose $\forall y \in Y$, $\exists$ a nbhd $\mathcal{O}_y$ of identity in $G = SL_3(\mathbb{R})$ s.t.

$$M_y \cap \mathcal{O}_y \subset HV$$

Suppose $\exists\ g_i \to e$ and $g_i \in M_y \setminus H$ (we will see that this is true as Hx is NOT closed.)

For $i \gg 1$, $g_i = h_i\, v(t_i) \Rightarrow S_0 \cdot (h_i v(t_i)) = S_0 + 2t_i S_1 \to S_0$

$\Rightarrow t_i \to 0$; $g_i y \in \overline{Hx} \Rightarrow v(t_i)\, y \in \overline{Hx}$ and $v(t_i) \to e$.

Claim 3
- $\exists\ Y' \subseteq \overline{Hx}$ which is B-minimal
- Y' is NOT a single B-orbit.

Exercise (About minimal sets) G: locally closed, s.c.

Ω: homogeneous G-space

$H \subseteq G$: closed subgroup

$X \subseteq \Omega$: H-minimal subset which is NOT an H-orbit

$\Rightarrow \forall\ x \in X$, $\exists g_i \in G \setminus H$ s.t.

① $g_i \to e$ ② $g_i x \in Hx$.

Using the above exercise for Y' we get that

$\exists g_i' \in G \setminus B$ s.t. ① $g_i' \to e$ ② $g_i' y \in By$

In particular $g_i' \in M_y \Rightarrow i \gg 1$, $g_i' = h_i' \, v(t_i')$.

and as above $h_i' \to e$ and $t_i' \to 0$.

$\Rightarrow \exists\ b_i \in B$ s.t. $g_i' y = b_i y \Rightarrow b_i^{-1} g_i' y = y$

$\Rightarrow \quad v(t_j)\, b_i^{-1}\, g_i'\, y = v(t_j)\, y$

$\Longrightarrow \quad b_i\, v(t_j)\, b_i^{-1} g_i'\, y = b_i\, v(t_j)\, y \in \overline{Hx}$.

And $\lim_{i\to\infty} \lim_{j\to\infty} b_i\, v(t_j)\, b_i^{-1} g_i' = e$

$\Rightarrow$ if $i,j \gg 1$, then $\underbrace{b_i\, v(t_j)\, b_i^{-1}}_{\substack{\cap \\ V}} g_i' \in HV$

(as A normalizes V and U commutes with V.)

$\Rightarrow$ for some t', $\quad v(t')\, h_i' \in HV$.

Claim 4. $v(t)\, h \in HV$ for some $h \in H$ and h close enough to e $\Rightarrow h \in B$.

So by the above claim and using $h_i' \to e$, for $i \gg 1$, $h_i' \in B$.

$\Rightarrow g_i' \in BV \setminus B$ and $b_i^{-1} g_i' \in \{ g \in BV \mid g y = y \} =: \Delta$

. Δ is a discrete subgroup of $BV = AUV$, and it contains

$\underbrace{b_i^{-1} h_i'}_{B}\, v(t_i')$, $t_i' \neq 0$ and $t_i' \to 0$.

Claim 5. $\Delta \subseteq UV$.

So $b_i^{-1} g_i' \in UV \Rightarrow b_i^{-1} h_i' \in UV \cap B = U$

$\Rightarrow v(t_i')\, y = h_i'\, b_i\, y \in Uy$

$$\left.\begin{aligned} &\Rightarrow v(t_i')\, Uy \subseteq Uy \\ &\overline{Uy} = Y \\ &\text{is } U\text{-minimal} \end{aligned}\right\} \Rightarrow v(t_i')\, Y = Y .$$

$\Rightarrow Y$ is invariant under $\overline{\langle v(t_i') \rangle} = V$ (as $t_i' \to 0$) .

$\Rightarrow Y$ is invariant under $\langle v(t_i') \rangle = V$ (as $t_i' \to 0$).

So again $Vy \subseteq \overline{Hx}$. ■

So it remains to prove the above Lemma and Claims.

Pf of Lemma. (Recall the setting: $\{u_t\} \subseteq GL(E)$ a unipotent flow.

$L := \{v \in E \mid u_t \cdot v = v\}$;

$M \subseteq E \setminus L$;

$p \in L \cap \overline{M}$;

$\Rightarrow \exists$ a non-constant poly. map ϕ s.t.

① $\phi(0) = p$ ② $\operatorname{Im} \phi \subseteq \overline{L \cap \{u_t\} M}$.)

W.L.O.G we can assume that $u_t = \exp(t \operatorname{diag}(J_{n_1}, \dots, J_{n_k}))$

where $J_{n_i} = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$ is an $n_i \times n_i$ Jordan block. So

$\vec{p} = \sum_{i=1}^{k} p_i \vec{e}_{1+n_{i-1}}$ where $n_0 = 0$, and $\exists \vec{v}_j = (\vec{v}_{j1}, \dots, \vec{v}_{jk}) \in M$

s.t. $\vec{v}_{j\ell} \to p_\ell \vec{e}_{1+n_{\ell-1}}$. Hence w.l.o.g. we can assume that

$k=1$, i.e. there is only one Jordan block.

$$\text{So } u_{st} \vec{v}_j = \sum_{k=0}^{n-1} \frac{s^k t^k}{k!} J^k \Big(\sum_{i=1}^{n} \alpha_i^{(j)} e_i\Big) = \sum_{k=0}^{n-1} \frac{s^k t^k}{k!} \sum_{i=k+1}^{n} \alpha_i^{(j)} e_{i-k}$$

$$= \sum_{0 \le k < i \le n} \frac{\alpha_i^{(j)} s^k}{k!} t^k e_{i-k} = \sum_{\ell=1}^{n} \Big(\sum_{k=0}^{n-\ell} \frac{\alpha_{\ell+k}^{(j)} s^k}{k!} t^k \Big) e_\ell .$$

Let $s_j := \max\Big\{ \Big(\frac{\alpha_{1+k}^{(j)}}{k!}\Big)^{-1/k} \Big| 1 \le k \le n-1 \Big\}$ (the coff. of the poly. coeff. of $\vec{e}_1$.). Then

- $\sum \frac{\alpha_{1+k}^{(j)}}{k!} s_j^k t^k \xrightarrow[j \to \infty]{} p(t)$ a non-constant poly.

- $\frac{\alpha_{\ell+k}^{(j)}}{k!} s_j^k \le s_j^{-\ell+1} \xrightarrow[j \to \infty]{} 0$ if $1 < \ell \le n$, which finishes the proof. ∎

Pf of Claim 1. (Recall. Any closed $\{u_t\}$-invariant subset of $\Omega^{(1)}(\mathbb{R}^n)$ has a minimal set. And such a set is compact.)

Suppose X is closed and $\{u_t\}$-invariant, using quantitative non-div. of unipotent flows, we found a closed subset X_0 of X and a compact set K s.t. $\forall\, x\in X_0$, $\frac{1}{T}|\{t\in[0,T] \mid u_t x\in K\}|\geq 1/2$ if $T\gg 1$ (or neg. direction)

Then by Zorn's lemma we concluded that $\exists\, Y\subseteq X_0$ which is $\{u_t\}$-minimal. Let K_1 be a compact nhbd of $Y\cap K$ in Y s.t.

$u_t\cdot K\subseteq K_1$ for any $t\in[-1,1]$. $\forall\, y\in Y, \exists\, t: u_t y\in K$

$\Rightarrow u_{\lfloor t\rfloor} y\in K_1 \Rightarrow Y=\bigcup_{n=0}^{\infty} u_{-n}K_1$ and $Y=\bigcup_{n=0}^{\infty} u_n K_1$. If Y is not compact, then for any n $\exists\, y_n\in Y$ s.t.

$u_{-i}\, y_n\notin K_1$ for $0\leq i<n$ and $y_n' := u_{-n}\, y_n\in K_1$.

Going to a subsequence $y_{n_k}'\to y\in K_1$. On the other hand, for any $m\in\mathbb{Z}^+$, $u_m\, y_{n_k}' = u_{m-n_k} y_{n_k}\notin K_1$, which implies $u_m y\notin K_1$.

Hence $u_1 y\notin \bigcup_{n=0}^{\infty} u_{-n}K_1$ which is a contradiction. ∎

Pf of claim 2. (Recall. $\rho: SL_3(\mathbb{R})\to GL(\underbrace{\{S\in M_3(\mathbb{R}) \mid S=S^t\}}_{E})$, $\rho(g)(S)=(g^{-1})^t S (g^{-1})$;

$L:=\{S\in E \mid \rho(u_t)(S)=S\}=\langle S_0, S_1\rangle$ and $L\cap SL_3(\mathbb{R})=S_0+\mathbb{R}S_1$.)

First we notice $S_0\in L$; and, if $S\in L$, then for some α,β

$S_{\alpha,\beta} := \alpha S+\beta S_0\in SL_3(\mathbb{R})\cap L$. Then $\underline{\forall t}$

$S_{\alpha,\beta} := \alpha S + \beta S_0 \in SL_3(\mathbb{R}) \cap L$. Then $\underline{\forall t}$

$$u(t)^{-1} S_{\alpha,\beta} (u(t)^{-1})^t = S_{\alpha,\beta} \Rightarrow S_{\alpha,\beta}^{-1} u(t) S_{\alpha,\beta} = u(-t)^t$$

(View S_0 as an element of the Weyl group, (almost) permutations)

$$S_0^{-1} S_{\alpha,\beta}^{-1} u(t) S_{\alpha,\beta} S_0 = S_0^{-1} (u(t)^t)^{-1} S_0$$
$$= u(t)$$

$\Leftrightarrow S_0 S_{\alpha,\beta} \in C_G(U) = UV \Rightarrow S_{\alpha,\beta} \in S_0 UV \cap$ symmetric

$$\begin{bmatrix} & & 1 \\ & -1 & \\ 1 & & \end{bmatrix}\begin{bmatrix} 1 & a & b \\ & 1 & a \\ & & 1 \end{bmatrix} = \begin{bmatrix} & & 1 \\ & -1 & -a \\ 1 & a & b \end{bmatrix} \text{ is symmetric} \Leftrightarrow a = 0$$

So $S_{\alpha,\beta} \in S_0 V \Rightarrow S \in \text{Span}(S_0, S_1)$ where $S_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Moreover $L \cap SL_3(\mathbb{R}) = \{ S_0 + t S_1 \mid t \in \mathbb{R} \}$. ∎

Pf of claim 3 (Recall. Any closed B-invariant subset of $\Omega^{(1)}(\mathbb{R}^n)$ has a B-minimal subset)

X: closed and B-invariant. Then to find U-minimal subset, we started with the following corollary of Quantitative Non-div.

$\forall \varepsilon > 0$ $\exists C_\varepsilon \subseteq \Omega^{(1)}(\mathbb{R}^n)$ compact s.t. for any unipotent flow $\{u_t\} \subseteq SL_n(\mathbb{R})$ and $x \in \Omega^{(1)}(\mathbb{R}^n)$ either

① $\underline{\lim} \frac{1}{T} |\{ t \in [0,T] \mid u_t x \in C_\varepsilon \}| \geq 1 - \varepsilon$

<u>or</u>

② $\exists\, \Delta \subseteq x$ a primitive proper subgroup s.t.

$d(u_t \Delta) \le \rho < 1$ and constant

in particular $u_t\, \Delta_{\mathbb{R}} = \Delta_{\mathbb{R}}$.

In our case the only proper subspaces of $\mathbb{R}^3$ that are invar. under $u_t = \begin{bmatrix} 1 & t & t^2/2 \\ & 1 & t \\ & & 1 \end{bmatrix}$ are $\mathbb{R}e_1$ and $\mathbb{R}e_1 \oplus \mathbb{R}e_2$.

So for any $x \in \Omega^{(1)}(\mathbb{R}^n)$ either $Ux \cap C_\varepsilon \ne \emptyset$ or

$d(x \cap \mathbb{Z}e_1) \le \rho$ or $d(x \cap \mathbb{Z}e_1 \oplus \mathbb{Z}e_2) \le \rho$.

In the latter, for $s \gg_x 1$, $d(a(s)x \cap \mathbb{Z}e_1) > 1$ and

$d(a(s)\, x \cap \mathbb{Z}e_1 \oplus \mathbb{Z}e_2) > 1$,

which implies $U a(s)\, x \cap C_\varepsilon \ne \emptyset$.

Hence $\forall x \in \Omega^{(1)}(\mathbb{R}^3)$, $Bx \cap C_\varepsilon \ne \emptyset$. Therefore by Zorn's lemma, there is a B-minimal subset of X.

Suppose Bx is closed. Then

$$B/B \cap g_0 SL_3(\mathbb{Z}) g_0^{-1} \longrightarrow Bx$$
$$b\Lambda \longmapsto bx$$

is a homeomorphism where $x = g_0 \mathbb{Z}^3$, $g_0 \in SL_3(\mathbb{R})$, and $\Lambda = B \cap g_0\, SL_3(\mathbb{Z})\, g_0^{-1}$.

Claim 6. Suppose $A = \mathbb{R}^+ \curvearrowright N$, N is an abelian top. group, and

$x \mapsto (a\cdot x)x^{-1}$ is an onto map for any $a \in A\setminus\{1\}$. Let $B = AKN$

Suppose $\forall x \in N$, $a\cdot x =: a x a^{-1} \xrightarrow[a\to 0^+]{} e$. And $\Lambda \subseteq B$ is a discrete subgroup. Then

either $\Lambda \subseteq N$ or $\Lambda = x \langle a_0 \rangle x^{-1}$ for some $x \in N$ and $a_0 \in A$.

By the above claim, either $\Lambda \subseteq U$ or $\Lambda = u_0 \langle a_0 \rangle u_0^{-1}$.

<u>First Case</u>: $s_i \to \infty$, then $a(s_i)\, u(t_i)\Lambda \to \infty$ in B/Λ as its projection to B/U does. Hence $a(s_i)\, u(t_i)\, x$ should escape to infinity. We have, however, that $a(s_i)\,u(t_i)\,x = u(s_i^2 t_i)\, a(s_i)\, x$ and we can choose t_i's so that $u(s_i^2 t_i)\, a(s_i) x \in C_\varepsilon$ if $s_i \gg 1$, which is a contradiction.

<u>Second Case</u>: Then $U a(s) \Lambda$ is closed for any s. And so

$$\begin{array}{ccc} t & \mapsto u(t)\, a(s)\,\Lambda \mapsto & u(t)\, a(s)\, x \\ \mathbb{R} & \longrightarrow & U a(s) x \end{array}$$

is a homeomorphism.

On the other hand, if s is large enough, $\{u(t)\, a(s)\, x\}_{t\in\mathbb{R}}$ does NOT escape to infinity. ■

<u>Pf of claim 4</u>. (<u>Recall</u>. $vh \in HV$ for $v \in V\setminus\{I\}$ and $h \in H$ close to I $\Rightarrow h \in B$.)

$$v(t)\,h = h'\,v(t') \Rightarrow S_0 \cdot v(t)h = S_0 \cdot v(t') = S_0 + 2t' S_1$$

$$\Rightarrow (S_0 + 2t\,S_1)\cdot h = S_0 + 2t' S_1$$

$$\Rightarrow S_0 \cdot h + 2t\, S_1 \cdot h = S_0 + 2t' S_1$$

$$\Rightarrow S_1 \cdot h = \frac{t'}{t} S_1$$

h close to I $\rightsquigarrow$ comes from $SL_2(\mathbb{R})$ $\rightsquigarrow$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} y & -2x \\ z & -y \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} * & * \\ \alpha z & * \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} ay+bz & -2ax-by \\ cy+dz & -2cx-dy \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} * & * \\ cdy+d^2z+2c^2x+cdy & * \end{bmatrix}$$

$$= \begin{bmatrix} * & * \\ \underbrace{d^2z+2c^2x+2cdy} & * \end{bmatrix}$$

$\hookrightarrow$ is a multiple of z $\Leftrightarrow$ $c=0$ $\Rightarrow$ $h \in B$ (close to I). ∎

Pf of Claim 5. (Recall. $\Delta \subseteq AUV$ and it contains a seq. $\{b_i v_i\}$ of elements s.t. $b_i \in AU$, $v_i \in V \setminus \{I\}$ and $v_i \longrightarrow I$. Then $\Delta \subseteq UV$.)

By claim 6, either $\Delta \subseteq UV$ or $\Delta = x_0 \langle a_0 \rangle x_0^{-1}$ for some $x_0 \in UV$ and $a_0 \in A$. In the second case,

$$u(t_0)\,v(t_0')\,a(s_0^n)\,v(-t_0')\,u(-t_0) = a(s^n)\,u(s_0^{-2n}t_0)\,v(s_0^{-4n}t_0')\,v(-t_0')\,u(-t_0)$$

$$= a(s^n)\,u((s_0^{-2n}-1)t_0)\,v((s_0^{-4n}-1)t_0').$$

So either the V component is I or it cannot converge to identity. ∎

abelian

Pf of claim 6

Pf of Claim 6. (Recall. $A = \mathbb{R}^+ \curvearrowright N$, $\forall x \in N$, $a^{-1} x a \xrightarrow[a\to\infty]{} e$.

$N \to N$, $x \mapsto (a \cdot x) x^{-1} = a x a^{-1} x^{-1}$ is onto for any $a \in A \setminus \{1\}$.

$\Lambda \subseteq A \ltimes N$ discrete subgroup. Then either $\Lambda \subseteq N$ or

$\Lambda = x_0 \langle a_0 \rangle x_0^{-1}$ for some $a_0 \in A$ and $x_0 \in N$.)

Suppose $\Lambda \not\subseteq N$. So $\exists\, ax \in \Lambda$ s.t. $a \neq 1 \Rightarrow \exists x' \in N$ s.t.

$$x' ax\, x'^{-1} = a\,(a^{-1} x' a x'^{-1})\, x = a.$$

$\Rightarrow$ $x' \Lambda x'^{-1}$ is a discrete subgroup of $A \ltimes N$

and it contains $a \in A \setminus \{1\}$.

if $a'u' \in x' \Lambda x'^{-1}$, then $a^{-n} a'u' a^n \longrightarrow a'$ $\Big\}\Rightarrow$
$x' \Lambda x'^{-1}$ is discrete

$n \gg 1$, $a^{-n} a'u' a^n = a' \Rightarrow u' = 1 \Rightarrow x' \Lambda x'^{-1} \subseteq A$

$\Rightarrow$ $x' \Lambda x'^{-1}$ is cyclic. ■