Exercise: Minkowski's theorems.

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. Let $\Lambda \in \Omega(\mathbb{R}^n)$ and $X \subseteq \mathbb{R}^n$ be a Lebesgue measurable set s.t. $\mathrm{vol}(X) > \mathrm{vol}(\mathbb{R}^n/\Lambda)$. Then

$$\exists\, x_1, x_2 \in X \text{ s.t. } x_1 - x_2 \in \Lambda.$$

<u>Solution</u> We know that $\Lambda = \bigoplus_{i=1}^{n} \mathbb{Z} v_i$ for some $v_i \in \Lambda$.

Let $\mathcal{F} = \left\{ \sum_{i=1}^{n} a_i v_i \,\middle|\, -\frac{1}{2} \leq a_i \leq \frac{1}{2} \right\}$. Then

① $\mathbb{R}^n = \Lambda + \mathcal{F}$.

② $\forall\, \lambda_1 \neq \lambda_2 \in \Lambda$, $\lambda_1 + \mathcal{F} \cap \lambda_2 + \mathcal{F} = \emptyset$.

$$\Rightarrow X = \bigsqcup_{\lambda \in \Lambda} (X \cap \lambda + \mathcal{F})$$

$$\Rightarrow \mathrm{vol}(X) = \sum_{\lambda \in \Lambda} \mathrm{vol}(X \cap \lambda + \mathcal{F})$$

$$= \sum_{\lambda \in \Lambda} \mathrm{vol}(-\lambda + X \cap \mathcal{F})$$

If $-\lambda_1 + X \cap -\lambda_2 + X = \emptyset$ for any $\lambda_1 \neq \lambda_2 \in \Lambda$, then

$$\mathrm{vol}(X) = \mathrm{vol}\left(\left(\bigcup_{\lambda \in \Lambda} (\lambda + X)\right) \cap \mathcal{F}\right) \leq \mathrm{vol}(\mathcal{F}) = \mathrm{vol}(\mathbb{R}^n/\Lambda)$$

which is a contradiction. ■

(Minkowski's convex body theorem) Let $C \subseteq \mathbb{R}^n$ be a Borel convex symmetric set. Suppose $\Lambda \in \Omega(\mathbb{R}^n)$ s.t.

$$\mathrm{vol}(C) > 2^n \, \mathrm{vol}(\mathbb{R}^n/\Lambda).$$

Then $|\Lambda \cap C| \geq 3$. (Symmetric means $C = -C$.)

Hint. Use the previous problem for $\frac{1}{2}C$.

- Prove that, $\forall \Lambda \in \Omega(\mathbb{R}^n)$, $\delta(\Lambda) \leq \sqrt{n}\left(\mathrm{vol}(\mathbb{R}^n/\Lambda)\right)^{\frac{1}{n}}$.

Hint. It is enough to find r s.t. $\mathrm{vol}(B(r)) > 2^n \mathrm{vol}(\mathbb{R}^n/\Lambda)$.

And volume of a ball of radius r in $\mathbb{R}^n \geq$

volume of the box $\left[\frac{-r}{\sqrt{n}}, \frac{r}{\sqrt{n}}\right]^n = \left(\frac{2r}{\sqrt{n}}\right)^n$.

Def. (Minkowski's successive minima).

Let $\Lambda \in \Omega(\mathbb{R}^n)$. For any $r \in \mathbb{R}^{>0}$, let

$$V_r := \mathbb{R}\text{-span of } (\Lambda \cap B(r)).$$

So the smallest r where $V_r \neq 0$ is $\delta(\Lambda)$.

For any $1 \leq i \leq n$, let $\lambda_i(\Lambda) := \inf\{r \mid \dim V_r \geq i\}$.

So $\lambda_1(\Lambda) = \delta(\Lambda)$. $\lambda_i(\Lambda)$ is called the i^{th} successive minima of Λ.

- Suppose $\Lambda \in \Omega(\mathbb{R}^n)$. From the definition of successive minima, we can get a sequence of vectors

$$v_1, v_2, \dots, v_n \in \Lambda$$

s.t. $\|v_k\| = \lambda_k(\Lambda)$ and $v_1, v_2, \dots, v_k$ are $\mathbb{R}$-linearly independent, for any $1 \le k \le n$.

- <u>(Minkowski's second theorem)</u> Let $\Lambda \in \Omega(\mathbb{R}^n)$.

Then $$\left(\lambda_1(\Lambda) \cdots \lambda_n(\Lambda)\right)^{1/n} \le \sqrt{n}\left(\mathrm{vol}(\mathbb{R}^n/\Lambda)\right)^{1/n}.$$

<u>Hint.</u> Let v_i's be as above, $V_k := \bigoplus_{i=1}^{k} \mathbb{R} v_i$, and $w_k = \mathrm{Pr}_{V_{k-1}^{\perp}}(v_k)$. Let C be the open ellipsoid with axis parallel to w_k's and length λ_k:

$$C := \left\{ \sum_{i=1}^{n} \frac{a_i}{\|w_i\|} w_i \;\middle|\; \sum_{i=1}^{n} \left(\frac{a_i}{\lambda_i}\right)^2 < 1 \right\}$$

$$= \left\{ v \;\middle|\; \sum \left(\frac{v \cdot w_i}{\lambda_i \|w_i\|}\right)^2 < 1 \right\}.$$

Show that $C \cap \Delta = \{0\}$. Conclude that

$$\mathrm{vol}(C) = \lambda_1 \cdot \ldots \cdot \lambda_n \cdot B(1) \leq 2^n \, \mathrm{vol}(\mathbb{R}^n/\Delta)$$

$$\Rightarrow \quad \left(\frac{2}{\sqrt{n}}\right)^n \lambda_1 \cdot \ldots \cdot \lambda_n \leq 2^n \, \mathrm{vol}(\mathbb{R}^n/\Delta).$$