Irrational Rotations

Wednesday, March 4, 2015 10:23 AM

Exercise $T_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, $T_\alpha(x) = x+\alpha$. Notice that $\forall \alpha$ The Lebesgue measure ℓ on $\mathbb{R}/\mathbb{Z}$ is T_α-invariant.

① Prove that ℓ is T_α-ergodic $\Leftrightarrow$ $\alpha \notin \mathbb{Q}$.

② Prove that ℓ is never T_α-mixing.

Hint. For ① use L^2-Fourier expansion.

For ②, if $\alpha \in \mathbb{Q}$, ℓ is NOT even T_α-ergodic. If $\alpha \notin \mathbb{Q}$, then $\exists\, n_i \to \infty$ s.t. $T_\alpha^{n_i} \sim id$.

Exercise $A \in M_n(\mathbb{Z})$ and $\det A \neq 0$. Then $T_A : (\mathbb{R}/\mathbb{Z})^n \to (\mathbb{R}/\mathbb{Z})^n$,

$$T_A([\vec{x}]) := [A\vec{x}]$$

is a well-defined surjective endomorphism which preserves the Lebesgue measure, i.e. $\ell(T_A^{-1} X) = \ell(X)$.

Prove that ℓ is T_A-ergodic $\Leftrightarrow$ no eigenvalue of A is a root of unity.